1 The abstract

In his epic text [3], Grothendieck reflected about deep relation between topos theory and homotopy theory, where he emphasised the importance of the sheaf theoretical objects corresponding to higher categorical structures. His motivation for introduction of such categorical structures, as (weak) n-categories and (weak) n-groupoids, was to provide algebraic models for homotopy n-types. The nonabelian cohomology provides invariants which classify such sheaf theoretical objects, which we call n-stacks. But the idea to study higher dimensional categories as coefficients for nonabelian cohomology, goes back to Roberts [4].

The aim of this thesis, is to give a complete description of this theory in the dimension $n = 2$. We define geometric objects which would be classified with 2-dimensional nonabelian cohomolgy, whose coefficients are given by the bigroupoid $\mathcal{B}$. We call such objects principal bigroupoid 2-bundles.

Bicategories were first defined by Benabou in [2], but to effectively deal with their actions, we needed to adopt Batanin's approach to higher (weak) categories given in [1]. For any internal bigroupoid $\mathcal{B}$ (given by the source and target funtors $T, S\colon B_1 \to D(B_0)$ from an internal groupoid B_1 to the discrete groupoid of the object X in $\mathcal{E}$) in the topos E (or even some Barr-exact category $\mathcal{E}$).

In order to get those objects, we first define the notion of the action of the bicategory. The action of the bicategory $\mathcal{B}$ is given by the pseudoalgebras over a pseudomonad naturally induced by $\mathcal{B}$.

Actually the more appropriate name for these objects would be $\mathcal{B}$-2-torsors. We will give several different characterizations of these geometric object and prove that they are all equivalent.

Definition 1.1. *The $\mathcal{B}$-2-bundle over an object X in $\mathcal{E}$ consists of the following data:*

- *an internal functor $p\colon \mathcal{P} \to X$ over X called the 2-bundle;*
- *an internal functor $m\colon \mathcal{P} \to D(B_0)$ (to the discrete groupoid of the object X in $\mathcal{E}$) called a momentum functor, together with*
- *an internal functor $a\colon \mathcal{P} \times_{B_0} B_1 \to \mathcal{P}$, (from the fibered product of groupoids $p\colon \mathcal{P} \to X$ and $T\colon B_1 \to D(B_0)$), called an action functor*

which satisfy the categorified versions of quassiassociativiy and unity of the action.

The $\mathcal{B}$-2-bundle is principal if the induced functor in Gpd(E)

$$(pr_1, a)\colon : \mathcal{P} \times_{B_0} B_1 \longrightarrow \mathcal{P} \times_X \mathcal{P}$$

is an equivalence over $\mathcal{P}$ in Gpd$(\mathcal{E})$.

By choosing its weak inverse, its second component $\mathcal{D} : P \times_X P \to B_1$ gives rise to the second nonabelian Čech cocycle with the values in the bigroupoid $\mathcal{B}$ in $\mathcal{E}$, by categorifing the usual methods, and for any object X in $\mathcal{E}$, this gives a biequivalence between (large) bigroupoids

$$2Tor(X, \mathcal{B}) \sim \mathcal{H}^2(X, \mathcal{B})$$

As the main result of the thesis, we extend results which goes back to Giraud, Duskin and Breen all of which comes from the ideas that are undoubtadely inherent in Grothendieck work, and prove them in the 2-dimensional context. One such theorem includes the extension of Duskin's (unpublished) result which states that for any gerbe $\mathcal{G}$ over $\mathcal{E}$, there exists an internal groupoid bouquet B in $Gpd(\mathcal{E})$, which he calls bouquet such that the gerbe can be recovered by torsors $Tors(B) \sim \mathcal{G}$. Its categorification is the follwing result:

Theorem 1.1. *Let $\mathcal{B}$ be an internal groupoid in the topos $\mathcal{E}$. The associated 2-stack to the small 2-fibration $\mathcal{FB}$ over $\mathcal{E}$ is the 2-stack $2Tors(\mathcal{B})$ of $\mathcal{B}$-2-torsors.*

Theorem 1.2. *Let $\mathcal{B}$ be an internal groupoid in the topos $\mathcal{E}$. The associated 2-stack to the small 2-fibration $\mathcal{FB}$ over $\mathcal{E}$ is the 2-stack $2Tors(\mathcal{B})$ of $\mathcal{B}$-2-torsors.*

Here the fibered 2-category $\mathcal{FB}$ over $\mathcal{E}$ comes as the "singular 2-functor" of the Yoneda embedding $y\colon \mathcal{E} \to Bgpd(\mathcal{E})$, so that its fiber over X in $\mathcal{E}$ is described by the small bigroupoid $\mathcal{FB}(X) = Bgpd(\mathcal{E})(y(X), \mathcal{B})$ For any covering $e\colon U \to X$, the restriction of this fibered bigrupoid to the simplicial resolution of e, defines a cosimplicial bigroupoid $\mathcal{CB}_\bullet$ and therefore defines bigroupoid $2Desc(\mathcal{CB}_\bullet)$ of its 2-descent data, which provides another biequivalence

$$2Tor(X, B) \sim_{bi} 2Desc(e)$$

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References

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