1 Actions of bicategories

Let $\mathcal{B}$ be a bicategory. There is a weak 2-monad $\mathcal{T}$ on a comma 2-category Cat $\downarrow \mathcal{B}_0$ naturally induced by $\mathcal{B}$ as following. It is a weak 2-functor $\mathcal{T}\colon \mathrm{Cat}\downarrow \mathcal{B}_0 \to \mathrm{Cat}\downarrow \mathcal{B}_0$, whose image for each object $\Lambda\colon \mathcal{C} \to \mathcal{B}_0$ of Cat $\downarrow \mathcal{B}_0$, is defined by $\mathcal{T}(\Lambda) := \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$.

Definition 1.1. *A right action of a bicategory $\mathcal{B}$ on a category $\mathcal{C}$ is given by by the following data:*

- *a functor $\Lambda\colon \mathcal{C} \to \mathcal{B}_0$ from the category $\mathcal{C}$ to the discrete category of objects $\mathcal{B}_0$ of the weak 2-category $\mathcal{B}$, called the momentum functor,*
- *a functor $\Phi\colon \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{C}$, called the action functor, and we usually write $\Phi(p, f) := p \triangleleft f$, for any object (p, f) in $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$, and $\Phi(a, \phi) := a \triangleleft \phi$ for any morphism $(a, \phi)\colon (p, f) \to (q, g)$ in $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$,*
- *a natural isomorphism*

$$\begin{tikzcd}
\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \arrow[r, "\Phi\times Id_{\mathcal{B}_1}"] \arrow[d, "Id_{\mathcal{C}}\times\mathcal{H}"'] & \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \arrow[d, "\Phi"] \arrow[dl, phantom, "\swArrow_{\kappa}"] \\
\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \arrow[r, "\Phi"'] & \mathcal{C}
\end{tikzcd}$$

whose component for any object (p, f, g) in $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$ is written as

$$\kappa_{p,f,g}\colon (p \triangleleft f) \triangleleft g \to p \triangleleft (f \circ g),$$

- *a natural isomorphism*

$$\begin{tikzcd}
\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_0 \arrow[r, "Id_{\mathcal{C}}\times I"] & \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \arrow[d, "\Phi"] \arrow[dl, phantom, "\swArrow_{\iota}"] \\
\mathcal{C} \arrow[u, "(Id_{\mathcal{C}}{,}\Lambda)"] \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

where we write for each object p in $\mathcal{C}$

$$\iota_p\colon p \triangleleft i_{\Lambda(p)} \to p$$

such that following axioms are satisfied:

- *equivariance of the action*

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \arrow[r, "\Phi"] \arrow[d, "Pr_1"'] & \mathcal{C} \arrow[d, "\Lambda"] \\ \mathcal{B}_1 \arrow[r, "D_1"'] & \mathcal{B}_0 \end{tikzcd}$$

which means that for any object (p, f) *in* $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$*, we have* $\Lambda(p \triangleleft f) = D_1(f)$*, and for any morphism* $(a, \phi)\colon (p, f) \to (q, g)$ *in* $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$*, we have* $\Lambda(a \triangleleft \phi) = D_1(\phi)$*,*

- *for any object* (p, f, g, h) *in* $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$ *the following diagram*

$$\begin{tikzcd} & ((p \triangleleft f) \triangleleft g) \triangleleft h \arrow[dl, "\kappa_{p,f,g} \triangleleft h"'] \arrow[dr, "\kappa_{p \triangleleft f, g, h}"] & \\ (p \triangleleft (f \circ g)) \triangleleft h \arrow[d, "\kappa_{p, f \circ g, h}"'] & & (p \triangleleft f) \triangleleft (g \circ h) \arrow[d, "\kappa_{p, f, g \circ h}"] \\ p \triangleleft ((f \circ g) \circ h) \arrow[rr, "p \triangleleft \alpha_{f,g,h}"'] & & p \triangleleft (f \circ (g \circ h)) \end{tikzcd}$$

commutes,

- *for any object* (p, f) *in* $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$ *following diagrams*

$$\begin{tikzcd} (p \triangleleft i_{\Lambda(p)}) \triangleleft f \arrow[rr, "\kappa_{p, i_{\Lambda(p)}, f}"] \arrow[dr, "\Phi \triangleleft f"'] & & p \triangleleft (i_{\Lambda(p)} \circ f) \arrow[dl, "p \triangleleft \lambda_f"] \\ & p \triangleleft f & \end{tikzcd} \qquad \begin{tikzcd} (p \triangleleft f) \triangleleft i_{S(p)} \arrow[rr, "\kappa_{p, f, i_{S(f)}}"] \arrow[dr, "\Phi \triangleleft i_{S(p)}"'] & & p \triangleleft (f \circ i_{S(p)}) \arrow[dl, "p \triangleleft \rho_f"] \\ & p \triangleleft f & \end{tikzcd}$$

commute.

Remark 1.1. *Note the fact that $\Phi\colon \mathcal{C}\times_{\mathcal{B}_0}\mathcal{B}_1 \to \mathcal{C}$ is a functor, immediately implies an interchange law*

$$(b \triangleleft \psi)(a \triangleleft \phi) = (ba) \triangleleft (\psi\phi)$$

Definition 1.2. *Let $(\mathcal{C}, \Lambda, \Phi, \alpha, \iota)$ and $(\mathcal{D}, \Psi, \Omega, \beta, \kappa)$ be two $\mathcal{B}$-categories. A $\mathcal{B}$-equivariant functor from $(\mathcal{C}, \Lambda, \Phi, \alpha, \iota)$ to $(\mathcal{D}, \Psi, \Omega, \beta, \kappa)$ is a pair (F, θ) consisting of*

- *a functor $F\colon \mathcal{C} \to \mathcal{D}$*
- *a natural transformation $\theta\colon F \circ \Psi \to \Phi \circ (F \times Id_{\mathcal{B}_1})$*

$$\begin{tikzcd}
\mathcal{C}\times_{\mathcal{B}_0}\mathcal{B}_1 \arrow[r, "F\times Id_{\mathcal{B}_1}"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\swArrow\,\theta"] & \mathcal{D}\times_{\mathcal{B}_0}\mathcal{B}_1 \arrow[d, "\Psi"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}$$

such that following conditions are satisfied

- $\Omega \circ F = \Lambda$

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "\Lambda"'] & & \mathcal{D} \arrow[dl, "\Omega"] \\
& \mathcal{B}_0 &
\end{tikzcd}$$

- *the diagram of natural transformations filling the faces of the cube*

$$\begin{tikzcd}
& \mathcal{C}\times_{\mathcal{B}_0}\mathcal{B}_1\times_{\mathcal{B}_0}\mathcal{B}_1 \arrow[rr, "\Phi\times Id_{\mathcal{B}_1}"] \arrow[dl, "Id_{\mathcal{C}}\times\mathcal{H}"'] \arrow[dd, "F\times Id_{\mathcal{B}_1\times_{\mathcal{B}_0}\mathcal{B}_1}"' near end] & & \mathcal{C}\times_{\mathcal{B}_0}\mathcal{B}_1 \arrow[dl, "\Phi"'] \arrow[dd, "F\times Id_{\mathcal{B}_1}"] \\
\mathcal{C}\times_{\mathcal{B}_0}\mathcal{B}_1 \arrow[rr, "\Phi" near end] \arrow[dd, "F\times Id_{\mathcal{B}_1}"'] & & \mathcal{C} \arrow[dd, "F" near start] & \\
& \mathcal{D}\times_{\mathcal{B}_0}\mathcal{B}_1\times_{\mathcal{B}_0}\mathcal{B}_1 \arrow[rr, "\Psi\times Id_{\mathcal{B}_1}" near start] \arrow[dl, "Id_{\mathcal{C}}\times\mathcal{H}"'] & & \mathcal{D}\times_{\mathcal{B}_0}\mathcal{B}_1 \arrow[dl, "\Psi"] \\
\mathcal{D}\times_{\mathcal{B}_0}\mathcal{B}_1 \arrow[rr, "\Psi"] & & \mathcal{D} &
\end{tikzcd}$$

commutes, which means that two natural transformations

$$(\theta \circ (Id_{\mathcal{C}} \times \mathcal{H}))(\Psi \circ (\theta \times Id_{\mathcal{B}_1}))(\beta \circ (F \times Id_{\mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1}))$$

and

$$(\Psi \circ (\theta \times Id_{\mathcal{B}_1}))(\theta \circ (\Phi \times Id_{\mathcal{B}_1}))\alpha$$

obtained by pasting, are equal.

- *the diagram of natural transformations, which fill the faces*

$$\begin{tikzcd}
& \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \arrow[rr, "F \times Id_{\mathcal{B}_1}"] \arrow[ddl, "\Phi"'] & & \mathcal{D} \times_{\mathcal{B}_0} \mathcal{B}_1 \arrow[ddl, "\Psi"] & \\
& & \mathcal{C} \arrow[ul, "Id_{\mathcal{C}} \times I\Lambda"'] \arrow[rr, "F"] & & \mathcal{D} \arrow[ul, "Id_{\mathcal{C}} \times I\Omega"'] \\
\mathcal{C} \arrow[urr, equal] \arrow[rr, "F"] & & \mathcal{D} \arrow[urr, equal] & &
\end{tikzcd}$$

commutes, meaning that we have an equation

$$(\Psi * id_L) \cdot (\theta * \Phi) \cdot (\kappa * F) = id_F * \iota$$

where $L\colon \mathcal{C} \to \mathcal{D} \times_{\mathcal{B}_0} \mathcal{B}_1$ is a functor given by the equality of functors $(F \times Id_{\mathcal{B}_1}) \circ (Id_{\mathcal{C}} \times I\Lambda) = I\Omega \circ F$, which, in turn follows from the equality $\Omega F = \Lambda$.

Definition 1.3. *A $\mathcal{B}$-equivariant natural transformation between $\mathcal{B}$-covariant functors $(F, \theta), (G, \zeta)\colon (\mathcal{C}, \Lambda, \Phi, \alpha, \iota) \to (\mathcal{D}, \Psi, \Omega, \beta, \kappa)$ is a natural transformation $\tau\colon F \to G$ such that following equality*

$$\begin{tikzcd}
\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \arrow[r, bend left, "F \times Id_{\mathcal{B}_1}"] \arrow[r, bend right, "G \times Id_{\mathcal{B}_1}"'] \arrow[r, phantom, "\Downarrow \tau \times \mathcal{B}1"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\Downarrow \theta"] & \mathcal{D} \times_{\mathcal{B}_0} \mathcal{B}_1 \arrow[d, "\Psi"] \\
\mathcal{C} \arrow[r, bend right, "G"'] & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \arrow[r, bend left, "G \times Id_{\mathcal{B}_1}"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\Downarrow \zeta"] & \mathcal{D} \times_{\mathcal{B}_0} \mathcal{B}_1 \arrow[d, "\Psi"] \\
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \tau"] & \mathcal{D}
\end{tikzcd}$$

of natural transformations is satisfied.

The above construction gives rise to the 2-category in an obvious way, so we have a following theorem.

Theorem 1.1. *The class of $\mathcal{B}$-categories, $\mathcal{B}$ equivariant functors and their natural transformations form a 2-category.*

Proof. The vertical and horizontal composition in a 2-category is induced from the composition in Cat. $\square$

2 Bigroupoid principal 2-bundles

Definition 2.1. *A right action of a bigroupoid $\mathcal{B}$ on a groupoid $\mathcal{P}$ is given by the action of the underlying bicategory $\mathcal{B}$ on a category $\mathcal{P}$ given as previously by $(\mathcal{P}, \mathcal{B}, \Lambda, \Phi, \alpha, \iota)$.*

Definition 2.2. *Let $\mathcal{B}$ be an internal bigroupoid in $\mathcal{E}$, and $\pi\colon \mathcal{P} \to X$ a bundle of groupoids over X in $\mathcal{E}$ on which $\mathcal{B}$ acts from the right. We say that $(\mathcal{P}, \pi, \Lambda, \mathcal{A}, X)$ is a right $\mathcal{B}$-principal 2-bundle (or a right $\mathcal{B}$-torsor) over X if the following conditions are satisfied:*

- *two canonical terminal morphisms $\pi_0\colon P_0 \to X$ and $\pi_1\colon P_1 \to X$ are epimorphisms,*
- *two canonical action morphisms $\lambda_0\colon P_0 \to B_0$ and $\lambda_1\colon P_1 \to B_0$ are epimorphisms,*
- *the induced internal functor*

$$(Pr_1, \Phi)\colon \mathcal{P} \times_{B_0} \mathcal{B}_1 \longrightarrow \mathcal{P} \times_X \mathcal{P}$$

  is a (strong) equivalence of internal groupoids over $\mathcal{P}$ (where both groupoids are seen as objects over $\mathcal{P}$ by the first projection functor).

Example 2.1. *(The unit principal 2-bundle) The unit $\mathcal{B}$-bundle is given by the triple $(\mathcal{B}_1, T, S, \mathcal{H}, B_0)$ where the momentum is given by the source functor $S\colon \mathcal{B}_1 \to \mathcal{B}_0$, and the action is given by the horizontal composition $\mathcal{H}\colon \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{B}_1$.*

Example 2.2. *(The pullback principal 2-bundle) For any principal $\mathcal{B}$-bundle $(\mathcal{P}, \pi, \Lambda, \Phi, X)$ over X, and any morphism $f\colon M \to B_0$, we have a pullback $\mathcal{B}$-principal bundle over M, defined as the quadruple $(f^*(\mathcal{P}), Pr_1, \Lambda \circ Pr_2, f^*(\Phi), M)$.*

3 Cocyclic description of principal 2-bundles

Since we assumed that the functor $(Pr_1, \Phi)\colon \mathcal{P} \times_{B_0} \mathcal{B}_1 \to \mathcal{P} \times_X \mathcal{P}$ is an equivalence, we choose its weak inverse

$$(\mathrm{Pr}_1, \mathcal{D})\colon \mathcal{P} \times_X \mathcal{P} \longrightarrow \mathcal{P} \times_{B_0} \mathcal{B}_1,$$

together with natural isomorphisms

$$(\mathrm{Pr}_1, \mu)\colon Id_{\mathcal{P} \times_{B_0} \mathcal{B}_1} \Longrightarrow (\mathrm{Pr}_1, \mathcal{D}) \circ (Pr_1, \Phi), \qquad (\mathrm{Pr}_1, \nu)\colon (Pr_1, \Phi) \circ (\mathrm{Pr}_1, \mathcal{D}) \Longrightarrow Id_{\mathcal{P} \times_X \mathcal{P}},$$

The second component of the above weak inverse is a functor $D\colon \mathcal{P} \times_X \mathcal{P} \longrightarrow \mathcal{B}_1$, which we we call a division functor, for reasons that we will soon explain. The component of the natural isomorphism $\nu\colon (Pr_1, \Phi) \circ (\mathrm{Pr}_1, \mathcal{D}) \Longrightarrow Id_{\mathcal{P} \times_X \mathcal{P}}$, indexed by the object $(p, q) \in \mathcal{P} \times_X \mathcal{P}$, is an isomorphism $\nu_{p,q}\colon p \triangleleft p^*q \to q$, where we use an abbreviation $p^*q := \mathcal{D}(p, q)\colon \lambda_0(q) \to \lambda_0(p)$, for the 1-morphism in $\mathcal{B}_1$. The natural isomorphism $\mu\colon Id_{\mathcal{P} \times_{B_0} \mathcal{B}_1} \Longrightarrow (\mathrm{Pr}_1, \mathcal{D}) \circ (Pr_1, \Phi)$, is indexed by the object $(p, f) \in \mathcal{P} \times_{B_0} \mathcal{B}_1$, by an isomorphism $\mu_{p,f}\colon p \to p^*(p \triangleleft f)$.

Let's now give a cocyclic description of the principal $\mathcal{G}$-bundle $\mathcal{P}$. Since the map $\pi\colon P_0 \to M$ is a surjective submersion, we can find an open cover $M = \bigcup U_i$ of the base manifold M together with local sections $\sigma_i\colon U_i \to P_0$ of the map π. The corresponding statement in the topos $\mathcal{E}$ is that epimorphism $\pi\colon P_0 \to M$ in $\mathcal{E}$ locally splits, since the diagonal morphism $\Delta\colon P_0 \to P_0 \times_M P_0$ is a splitting in $\mathcal{E}/P_0$ of the pullback bundle $\pi^*(\pi)\colon P_0 \times_M P_0 \to P_0$ which is given by $pr_1\colon P_0 \times_M P_0 \to P_0$.

We use the division functor to define $g_{ij} = \mathcal{D}(\sigma_i, \sigma_j)\colon U_{ij} \to B_1$, and a local sections $f_{ij}^{\alpha}\colon U_{ij}^{\alpha} \to P_1$ of $\pi s = \pi t$ over some covering U_{ij}^{α} of U_{ij} such that

$$f_{ij}\colon \sigma_j \to \sigma_i \triangleleft g_{ij}.$$

The following diagram

$$\begin{tikzcd}
\sigma_k \arrow[r, "f_{jk}"] \arrow[dd, "f_{ik}"'] & \sigma_j \triangleleft g_{jk} \arrow[d, "f_{ij} \triangleleft g_{jk}"] \\
& (\sigma_i \triangleleft g_{ij}) \triangleleft g_{jk} \arrow[d, "\kappa_{ijk}"] \\
\sigma_i \triangleleft g_{ik} \arrow[r, "\sigma_i \triangleleft \beta_{ijk}"'] & \sigma_i \triangleleft (g_{ij} \circ g_{jk})
\end{tikzcd}$$

defines a morphism in $\psi \in Hom_{\mathcal{P}\times_X\mathcal{P}}(\sigma_i \lhd g_{ik}, \sigma_i \lhd (g_{ij} \circ g_{jk}))$ by the composition

$$\sigma_i \lhd g_{ik} \xrightarrow{f_{ik}^{-1}} \sigma_k \xrightarrow{f_{jk}} \sigma_j \lhd g_{jk} \xrightarrow{f_{ij} \lhd g_{jk}} (\sigma_i \lhd g_{ij}) \lhd g_{jk} \xrightarrow{\kappa_{ijk}} \sigma_i \lhd (g_{ij} \circ g_{jk})$$

and since the set $Hom_{\mathcal{P}\times_X\mathcal{P}}(\sigma_i \lhd g_{ik}, \sigma_i \lhd (g_{ij} \circ g_{jk}))$ is an image of the induced functor (Pr_1, Φ) which defines a bijective correspondence with the set $Hom_{\mathcal{P}\times_{\mathcal{B}_0}\mathcal{B}_1}((\sigma_i, g_{ik}), (\sigma_i, g_{ij} \circ g_{jk}))$ the inverse image of ψ defines sections $\beta_{ijk} \colon g_{ik} \to g_{ij} \circ g_{jk}$ in B_2, such that the diagram becomes the identity

$$(\sigma_i \lhd \beta_{ijk}) f_{ik} = \kappa_{ijk} (f_{ij} \lhd g_{jk}) f_{jk}$$

Theorem 3.1. *Any $\mathcal{B}$-2-torsor $\pi \colon \mathcal{P} \to X$ gives rise to the class $\mathcal{H}^2(X, \mathcal{B})$.*

Proof. Consider the following cube

$$\begin{tikzcd}
 & \sigma_l \arrow[rrr, "f_{kl}"] \arrow[ddl, "f_{jl}"'] \arrow[dddd, dashed, "f_{il}"] & & & \sigma_k \lhd g_{kl} \arrow[dl, "f_{jk} \lhd g_{kl}"'] \arrow[ddd, "f_{ik} \lhd g_{kl}"] \\
 & & & (\sigma_j \lhd g_{jk}) \lhd g_{kl} \arrow[dl, "\kappa_{j,jk,kl}"'] \arrow[d, "(f_{ij} \lhd g_{jk}) \lhd g_{kl}"] & \\
\sigma_j \lhd g_{jl} \arrow[rr, "\sigma_j \lhd \beta_{jkl}"] \arrow[dddd, "f_{ij} \lhd g_{jk}"'] & & \sigma_j \lhd (g_{jk} \circ g_{kl}) \arrow[d, "f_{ij} \lhd (g_{jk} \circ g_{kl})"'] & ((\sigma_i \lhd g_{ij}) \lhd g_{jk}) \lhd g_{kl} \arrow[dl, "(f_{ij} \lhd g_{jk}) \lhd g_{kl}"] \arrow[dd, "\kappa_{i,ijk,kl} \lhd g_{kl}"] & \\
 & & (\sigma_i \lhd g_{ij}) \lhd (g_{jk} \circ g_{kl}) \arrow[ddd, "\kappa_{i,ij,jkl}"'] & & (\sigma_i \lhd g_{ik}) \lhd g_{kl} \arrow[dl, "(\sigma_i \lhd \beta_{ikl}) \lhd g_{kl}"'] \arrow[d, "\kappa_{i,ik,kl}"] \\
 & \sigma_i \lhd g_{il} \arrow[rr, dash, dashed, "\sigma_i \lhd \beta_{ikl}"] \arrow[ddl, dashed, "\sigma_i \lhd \beta_{ijl}"'] & & (\sigma_i \lhd (g_{ij} \circ g_{jk})) \lhd g_{kl} \arrow[r, dash] \arrow[d, "\kappa_{i,ij,kl}"] & \sigma_i \lhd (g_{ik} \lhd g_{kl}) \arrow[dl, "\sigma_i \lhd (\beta_{ikl} \circ g_{kl})"] \\
 & & & \sigma_i \lhd ((g_{ij} \circ g_{jk}) \circ g_{kl}) \arrow[dl, "\sigma_i \lhd \alpha_{ij,jk,kl}"] & \\
\sigma_i \lhd (g_{ij} \lhd g_{jl}) \arrow[rr, "\sigma_i \lhd (g_{ij} \circ \beta_{ijk})"'] & & \sigma_i \lhd (g_{ij} \circ (g_{jk} \circ g_{kl})) & &
\end{tikzcd}$$

in which all faces except the bottom and right faces are diagrams which define nonabelian cocycles. The right face consists of one such diagram acted by g_{kl}, two are instances of naturality of the action, and one is coherence for action. Since these five faces of the cube in which all arrows are invertible commute, it follows that the sixth (bottom) face

$$
\begin{tikzcd}
\sigma_i \lhd g_{il} \arrow[r, "\sigma_i \lhd \beta_{ikl}"] \arrow[dd, "\sigma_i \lhd \beta_{ijl}"'] & \sigma_i \lhd (g_{ik} \circ g_{kl}) \arrow[d, "\sigma_i \lhd (\beta_{ijk} \circ g_{kl})"] \\
 & \sigma_i \lhd ((g_{ij} \circ g_{jk}) \circ g_{kl}) \arrow[d, "\sigma_i \lhd \alpha_{ij,jk,kl}"] \\
\sigma_i \lhd (g_{ij} \circ g_{jl}) \arrow[r, "\sigma_i \lhd (g_{ij} \circ \beta_{ijk})"'] & \sigma_i \lhd (g_{ij} \circ (g_{jk} \circ g_{kl}))
\end{tikzcd}
$$

also commutes. Since the functor $(Pr_1, \Phi)\colon \mathcal{P} \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{P} \times_X \mathcal{P}$ is fully faithful, the inverse image of the diagonal 2-morphism from $\sigma_i \lhd g_{il}$ to $\sigma_i \lhd (g_{ij} \circ (g_{jk} \circ g_{kl}))$ in the above diagram, consists of the single 2-morphism between g_{il} and $(g_{ij} \circ (g_{jk} \circ g_{kl}))$ which gives the identity

$$(g_{ij} \circ \beta_{jkl})\beta_{ijl} = \alpha_{ij,jk,kl}(\beta_{ijk} \circ g_{kl})\beta_{ikl}$$

for the nonabelian 2-cocycle (g_{ij}, β_{ijk}) with values in the bigroupoid $\mathcal{B}$. □

Theorem 3.2. *The above correspondence gives a biequivalence*

$$2Tors(X, \mathcal{B}) \sim_{bi} \mathcal{H}^2(X, \mathcal{B})$$

Theorem 3.3. *There exist a biequivalence*

$$2Tors(X, \mathcal{B}) \sim_{bi} Bun(\mathcal{B})$$

Proof. Let's again choose local sections $\sigma_i\colon U_i \to P_0$ of a surjective submersion $\pi\colon P_0 \to M$, and we consider the morphism $\tau\colon U \to B_0$, defined by $\tau = (\tau_i)_{i \in I}$, where $U = \coprod_{i \in I} U_i$ and $\tau_i := \lambda_0 \sigma_i\colon U_i \to B_0$. Than the induced morphism

$$\phi\colon \tau^*(\mathcal{B}_1) \longrightarrow \mathcal{P}|_U$$

defined by $\phi(x_i, g) = \sigma_i(x_i)g$ is an equivalence since it is an equivariant morphism of $\mathcal{B}$-2-torsors, and the 2-category $2Tors(X, \mathcal{B})$ is a bigroupoid. □