On the BV-formalism

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Abstract

- We try to **understand the Batalin-Vilkovisky (BV) complex** for handling perturbative quantum field theory.
- I emphasize a **Lie ∞-algebraic perspective** based on [Roberts-S., Sati-S.-Stasheff] over the popular supergeometry perspective and try to show how that is useful.
- A couple of **examples** are spelled out in detail:
  - the (-1)-brane;
  - ordinary gauge theory;
  - higher gauge theory.
- Using these we demonstrate that the BV-formalism arises naturally from a construction of **configuration space from an internal hom-object** following in spirit, but not in detail, the very insightful [AKSZ, Roytenberg].

Thanks

While banging my head against the BV-formalism, I profited from conversation with

- John Baez
- Dmitry Roytenberg
- Pavol Ševera
- Jim Stasheff
- Danny Stevenson
- Zoran Škoda
- Todd Trimble

Of course none of them can be taken responsible for what is wrong and/or idiosyncratic about my presentation, as far as it is.
The discussion of the higher gauge theory example is based on [Baez-S., S.-Waldorf, Sati-S.-Stasheff].

Plan

Introduction

next: ∞-vector spaces

The setup of field theory

Classical field theory.

- a "space" of configurations

$$\mathrm{conf}$$

- an "action functional"

$$\exp(S) : \mathrm{conf} \to \mathrm{U}(1)$$

Quantum field theory.

- something like a measure

$$d\mu$$

  on conf
- the path integral

$$\int_{\mathrm{conf}} \exp(S)\, d\mu$$

The problem and one of its solutions

Problem. conf is usually "two large" in two ways:

- conf is typically not a finite dimensional manifold;
- conf typically has many flows along which $\exp(S)$ is invariant, the *symmetries*.

One solution: perturbative quantum field theory

- Taylor-expand S around critical points of S (the *classical solutions*);
- inject conf into a supermanifold and extend $\exp(S)$ to a superfunction such that the odd part of the path integral (over the *ghosts*) divides out the symmetries.

Tools to take care of that

- **Dealing with the symmetries.**
  The **BRST-complex** is the Chevalley-Eilenberg complex of the symmetries acting on the module of functions on conf.
  In **0-th cohomology**: the functions invariant under the symmetries.
- **Dealing with the critical points.**
  The **Koszul-Tate complex** is the *weak quotient* of all functions on conf by those that vanish on the critical points of S.
  In **0-th cohomology**: the functions on the critical points of S.

BV-complex

The BRST complex and the Koszul-Tate complex happen to **unify** to the **Batalin-Vilkovisky** (BV) complex

$$\mathbf{BV}^\bullet(\mathrm{conf}, \exp(S)) .$$

If we think of conf as a **supermanifold**, then

- the BV complex is the space of functions the shifted **cotangent bundle** of conf, regarded as a supermanifold

$$\mathrm{BV}^\bullet = C^\infty(T[1]^*\mathrm{conf}) .$$

If we think of conf as a **Lie *n*-algebroid**, then

- the BV complex is the space of horizontal inner derivations on the **Weil algebra** of conf

$$\mathrm{BV}^\bullet = \mathrm{horinn}(\mathrm{W}(\mathrm{conf})) .$$

It is useful to regard the BV-formalism as being about Lie ∞-algebroids:

Sketches of the ∞-Lie theorem

Lie's classical theorem says that every Lie algebra is the linear approximation to some Lie group.
People are progressing towards the many-object (Lie algebroid) and higher dimensional (Lie n-algebroids) version of this:

∞-Lie theorem

$$\begin{tikzcd}
L_\infty\text{-algebra} \arrow[rr, "\text{[Sullivan], [Getzler], [Henriques]}"] \arrow[d, hook] & & \text{Lie } \infty\text{-groups} \arrow[d, hook] \\
\text{Lie } \infty\text{-algebroids} \arrow[d, dash, ":="'] & & \text{Lie } \infty\text{-groupoids} \arrow[ll, "\text{[Ševera]}"'] \arrow[d, dash, ":="] \\
\text{dg-manifolds} \arrow[rr, leftrightarrow] & & \text{smooth Kan-complexes}
\end{tikzcd}$$

Underlying this are two classical results:

Fact (Dold-Kan, Brown-Higgins)

	degree	$+$	0	-
Chain complexes are ∞-vector spaces.	**interpretation**	ordinary vector space		

$$\underbrace{}_{\infty-covector\ space} \qquad \underbrace{}_{\infty-\text{vector space}}$$

Fact (Sullivan models in rational homotopy theory)

Every non-negatively graded commutative differential A algebra is essentially the algebra of differential forms on *some* "space" X_A

$$A \hookrightarrow \Omega^\bullet(X_A)\,.$$

How to interpret the BV-complex

Using the ∞-Lie theorem, we can read the BV-formalism as saying

Observation

The configuration "space" of a physical system is really

- a Lie ∞-groupoid:
  - objects are the physical configurations;
  - morphisms are the gauge transformations between configurations;
  - 2-morphisms are the gauge-of-gauge transformations;
  - n-morphisms are the n-fold gauge transformations.

The Lie ∞-algebraic interpretation of the BV-complex

	BV-terminology	**DGCA interpretation**	**Lie ∞-groupoid interpretation**
fields and ghosts	fields	degree 0 generators of $\mathrm{CE}(\mathfrak{g}, V)$	objects of configuration space
	ghosts	degree 1 generators of $\mathrm{CE}(\mathfrak{g}, V)$	morphisms in configuration space
	n-fold ghosts	degree n generators of $\mathrm{CE}(\mathfrak{g}, V)$	n-morphisms in configuration space
antifields and antighosts	antifields	degree 1 horizontal derivations in $\mathrm{W}(\mathfrak{g}, V)$	paths of objects
	antighosts	degree 2 horizontal derivations in $\mathrm{W}(\mathfrak{g}, V)$	paths of 1-morphisms
	anti-ghosts-of-ghosts	degree 3 horizontal derivations in $\mathrm{W}(\mathfrak{g}, V)$	paths of 2-morphisms

What is the relation between the BV complex and the Weil algebra of a Lie ∞-algebroid?

Lie ∞-algebras

A L_∞-algebra $\mathfrak{g}$ – the infinite categorification of a Lie algebra – is a codifferential graded co-commutative coalgebra

$$\mathfrak{g} = (\vee^\bullet V, D_{\mathfrak{g}})$$

whose underlying graded co-commutative coalgebra is freely generated from a $\mathbb{N}_+$-graded vector space V.

Chevalley-Eilenberg algebras of Lie ∞-algebras

The Chevalley-Eileneberg complex corresponding to any Lie ∞-algebra $\mathfrak{g}$ is the complex

$$\cdots \quad \mathrm{Alt}(\mathfrak{g}^{\otimes(n-1)}, \mathbb{R}) \xrightarrow{D_{\mathfrak{g}}^*} \mathrm{Alt}(\mathfrak{g}^{\otimes n}, \mathbb{R}) \xrightarrow{D_{\mathfrak{g}}^*} \mathrm{Alt}(\mathfrak{g}^{\otimes(n+1)}, \mathbb{R}) \to \cdots \; .$$

In the case that $\mathfrak{g}$ is finite dimensional, this is a differential graded-commutative algebra

$$\mathrm{CE}(\mathfrak{g}) := (\wedge^\bullet V^*, d_{\mathrm{CE}(\mathfrak{g})})$$

with

$$d_{\mathrm{CE}(\mathfrak{g})} := D_{\mathfrak{g}}^* \,,$$

whose underlying graded commutative algebra is free.

In fact, every differential graded-commutative algebra DGCA whose underlying graded commutative algebra is free on a finite dimensional $\mathbb{N}_+$-graded vector space comes from a finite dimensional L_∞-algebra this way.

Weil algebra

There is a generalization of the construction of a *mapping cone* from complexes to dg-algebras.
First consider:

Definition (normal L_∞-subalgebra)

We say a Lie ∞-algebra $\mathfrak{h}$ is a normal sub L_∞-algebra of the L_∞-algebra $\mathfrak{g}$ if there is a morphism

$$\mathrm{CE}(\mathfrak{h}) \overset{t^*}{\twoheadleftarrow} \mathrm{CE}(\mathfrak{g})$$

which the property that

- on $\mathfrak{g}^*$ it restricts to a surjective linear map $\mathfrak{h}^* \overset{t_1^*}{\twoheadleftarrow} \mathfrak{g}^*$;
- if $a \in \ker(t^*)$ then $d_{\mathrm{CE}(\mathfrak{g})} a \in \wedge^\bullet(\ker(t_1^*))$.

Proposition

For $\mathfrak{h}$ and $\mathfrak{g}$ ordinary Lie algebras, the above notion of normal sub L_∞-algebra coincides with the standard notion of normal sub Lie algebras.

Weil algebra

Definition (mapping cone of qDGCAs; crossed module of normal sub L_∞-algebras)

Let $t : \mathfrak{h} \hookrightarrow \mathfrak{g}$ be an inclusion of a normal sub L_∞-algebra $\mathfrak{h}$ into $\mathfrak{g}$. The mapping cone of t^* is the qDGCA whose underlying graded algebra is

$$\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{h}^*[1])$$

and whose differential d_t is such that it acts on generators schematically as

$$d_t = \begin{pmatrix} d_{\mathfrak{g}} & 0 \\ t^* & d_{\mathfrak{h}} \end{pmatrix} .$$

The mapping cone L_∞-algebra thus obtained we denote

$$(\mathfrak{h} \stackrel{t}{\hookrightarrow} \mathfrak{g}) .$$

Definition: inner derivations and Weil algebra

For any L_∞-algebra $\mathfrak{g}$, its L_∞-algebra of inner derivation is the mapping cone on the identity of $\mathfrak{g}$

$$\mathrm{inn}(\mathfrak{g}) := (\mathfrak{g} \xrightarrow{=} \mathfrak{g}) .$$

The Weil algebra $\mathrm{W}(\mathfrak{g})$ of $\mathfrak{g}$ is the Chevalley-Eilenberg algebra of of $\mathrm{inn}(\mathfrak{g})$:

$$\mathrm{W}(\mathfrak{g}) := \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) .$$

The Weil algebra turns out to be trivializable, and yet, partly because of that, it is of crucial importance.

Observation (Weil algebra and shifted tangent bundle

If we regard $\mathrm{CE}(\mathfrak{g})$ from the point of view of supermanifolds as the algebra of functions on a supermanifold $X_{\mathfrak{g}}$, then $\mathrm{W}(\mathfrak{g})$ is the algebra of functions on the shifted tangent bundle $T[1]X_{\mathfrak{g}}$ of $X_{\mathfrak{g}}$:

$$\mathrm{W}(\mathfrak{g}) = C^{\infty}(T[1]X_{\mathfrak{g}}) .$$

Recall that the BV-complex can be thought of as the shifted *co*tangent bundle of a supermanifold.

tangent category	inner automorphism $(n+1)$-group	inner derivation Lie $(n+1)$-algebra	Weil algebra	shifted tangent bundle

$$\mathrm{CE}(\mathrm{Lie}(T\mathbf{B}G)) = \mathrm{CE}(\mathrm{Lie}(\mathrm{INN}(G))) = \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) = \mathrm{W}(\mathfrak{g}) = C^\infty(T[1]X_{\mathfrak{g}})$$

Figure: **A remarkable coincidence of concepts** relates the notion of graded tangency to the notion of universal bundles. See [Roberts-S.] and [Sati-S.-Stasheff].

With that nice interpretation in hand, the question remains how we extract from a given physical setup the relevant configuration Lie ∞-groupoid such that its Lie n-algebroid supports a BV-complex. In [AKSZ, Roytenberg] it was observed, more or less explicitly, that the entire BV-setup drops out, more or less automatically, if one forms the configuration space object (for field theories of general σ-model type) as the *internal hom-object* of maps from parameter space to configuration space.
We here adopt that general idea, but give a possibly different, systematic and conceptual, construction of the internal hom-object.

Internalized physics

For field theories of σ-model and of gauge theory type, we are dealing with objects

- par – **parameter space**
- tar – **target space**
- phas – **space of phases**

in some ambient category T, together with a morphism

- $\mathrm{tra} : \mathrm{tar} \to \mathrm{phas}$ – the **background field**

which we want to **transgress** to

- $\mathrm{conf} := \mathrm{hom}(\mathrm{par}, \mathrm{tar})$ – the **configuration space** of fields ,

which is the internal hom-object of maps from par to tar in T.

Internalized physics

The situation is a diagram

$$
\begin{tikzcd}
 & \mathrm{maps}(\mathrm{par}, \mathrm{tar}) \otimes \mathrm{par} \arrow[dl] \arrow[dr, "\mathrm{ev}"] & & \\
\mathrm{par} & & \mathrm{tar} \arrow[r, "\mathrm{tra}"] & \mathrm{phas}
\end{tikzcd}
$$

in T from which we build the **action functional**

$$\exp(S) : 1 \to \mathrm{hom}(\mathrm{conf}, \mathrm{maps}(\mathrm{par}, \mathrm{tar}))\,.$$

Example for internalized physics: ordinary gauge theory

A good concrete example of this to keep in mind is the example of ordinary gauge theory of $\mathfrak{g}$-connections on a manifold Y, which we discuss in detail in Example: ordinary gauge theory.
Here we are working with T the category of diferential graded-commutative algebras of non-negative degree. We set

Internalization of ordinary gauge theory

- $\mathrm{par} = \Omega^\bullet(Y)$ – the algebra of forms on Y
- $\mathrm{tar} = \mathrm{W}(\mathfrak{g})$ – the Weil algebra of $\mathfrak{g}$
- $\mathrm{conf} = \Omega^\bullet(U \mapsto \mathrm{Hom}(\mathrm{W}(\mathfrak{g})), \Omega^\bullet(Y \times U))$ – the algebra of forms on the presheaf on manifolds (i.e. the generalized smooth space) of dg-morphisms from $\mathrm{W}(\mathfrak{g})$ to $\Omega^\bullet(Y)$.

Example for internalized physics: ordinary gauge theory

This specifies the kinematics of ordinary gauge theory. We demonstrate in Example: ordinary gauge theory that the configuration space object thus obtained does indeed support the BV complex in that it knows about all the fields, ghosts, antifields and antighosts.

To do dynamics, we need to choose a "background field" $\mathrm{tra} : \mathrm{tar} \to \mathrm{phas}$. A good example to keep in mind for this is **topological Yang-Mills theory** with action functional

$$(A \in \Omega^\bullet(Y, \mathfrak{g})) \mapsto \int_Y \langle F_A \wedge F_A \rangle \,.$$

Example for internalized physics: ordinary gauge theory

This is obtained by taking

- $\mathrm{phas} = \mathrm{CE}(b^3\mathfrak{u}(1))$ – the Lie 4-algebra triply shifted $\mathfrak{u}(1)$
- $(\ \mathrm{tar} \xleftarrow{\mathrm{tra}} \mathrm{phas}\ ) = (\ \mathrm{W}(\mathfrak{g}) \xleftarrow{\langle\cdot,\cdot\rangle} \mathrm{CE}(b^3\mathfrak{u}(1))\ )$ – the embedding of the invariant polynomial $\langle\cdot,\cdot\rangle$ into $\mathrm{W}(\mathfrak{g})$.

Here I am making extensive use of the stuff discussed in Sati-S.-Stasheff, which eventually I need to review here.
Notice the reversal of the above arrows due to the fact that with algebras everything is contravariant.

previous: Introduction

∞-Vector spaces: (co)chain complexes

- A-modules
- Chain complexes as internal ω-categories
- Cochain complexes of A-modules

next: Lie ∞-algebras

The objects that we shall be concerned with here are **differential graded algebras** (dg-algebras). The right way to think of a dg-algebra is as a monoid in a category of cochain complexes.

dg-algebra	**monoid in $\mathbf{Ch}^\bullet(A)$**
wedge product	tensor product
graded-commutativity	nontrivial symmetric braiding

Table: **Realizing dg-algebras as monoids in chain complexes**.

We recall very basic facts about the category of chain complexes and the homological algebra one can do with it.

A-modules

A module is for an algebra precisely what a representation is for a group.

Definition

For A any algebra over k, a **module** N for A is a k-vector space N together with an action of A on N by linear operators, namely an algebra homomorphism

$$\rho : A \to \mathrm{End}_{\mathrm{Vect}_k}(N)\,.$$

Examples.

- A module for the ground field k regarded as an algebra over itself is nothing but an ordinary k-vector space.
- A module for the polynomial algebra $k[X]$ over a single variable is a vector space with one singled out endomorphism $\rho(X) \in \mathrm{End}(N)$ of it.
- The space of sections of a k-vector bundle $E \to X$ over some space X is a module over the algebra of k-valued functions on X.

The last example is the crucial one in the context of the BV formalism. Therefore we recall the statement underlying it in full detail.

Definition (special properties of modules)

The following special properties of A-modules are important.

- An A-module N is **finitely generated** if it is spanned, over A, by finitely many of its elements.
- An A-module N is **free** of rank $n \in \mathbb{N}$ if it is of the form $N \simeq A^n := A^{\oplus n}$.
- An A-module N is **projective** if any of the following equivalent conditions hold
  - N is a **direct summand of a free module**, i.e. there exists another module N' such that $N \oplus N'$ is free.
  - N is the image $N \simeq \mathrm{im}(P)$ of a projection $P \in \mathrm{End}(A^n)$ on some free module.
  - N satisfies the lifting property $\forall g, f$:

$$\begin{tikzcd} & M' \arrow[d] \\ N \arrow[ur, dotted, "\exists h"] \arrow[r, "g"'] & M \end{tikzcd}$$

Swan's theorem

For X a real manifold and $A = C(X)$ the algebra of real functions on X, the sections of vector bundles on X are precisely the finitely-generated projective modules over $A = C(X)$:

$$\mathrm{VectBun}(X) \stackrel{\simeq}{\to} A\mathrm{Mod}_{\mathrm{fin,proj}}$$

$$(E \to X) \mapsto \Gamma(E) .$$

We will come back to this special case.

There is an obvious notion of homomorphisms of A-modules.

Definition

Given two A-modules N and N', an A-module homomorphism

$$f : N \to N'$$

between them is a linear map that preseves the A action:

$$\begin{tikzcd}
A \otimes N \arrow[r, "\rho \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id}_A \otimes f"] & \mathrm{End}(N) \otimes N \arrow[r, "\mathrm{ev}"] & N \arrow[d, "f"] \\
A \otimes N' \arrow[r, "\rho' \otimes \mathrm{Id}"] & \mathrm{End}(N') \otimes N' \arrow[r, "\mathrm{ev}"] & N'
\end{tikzcd} .$$

Definition (the category of A-modules)

We write $A\mathrm{Mod}$ for the category whose objects are A-modules and whose morphisms are A-module homomorphisms.

Structures and properties of $A\mathrm{Mod}$.

- $A\mathrm{Mod}$ is an **abelian category**, hence we can do homological algebra inside $A\mathrm{Mod}$.
  Recall that this is equivalent to saying that that
  - it has a zero-object – this is the 0-dimensional A-module;
  - and it has all pullbacks and pushouts;
  - and all monomorphisms and epimorphisms are normal.
- $A\mathrm{Mod}$ is **symmetric monoidal**. The tensor product

$$\otimes_A : A\mathrm{Mod} \times A\mathrm{Mod} \to A\mathrm{Mod}$$

  is the ordinary tensor product of A-modules over A. The **tensor unit** is $I = A$ and the **symmetric braiding** is the obious one.

Structures and properties of $A\mathrm{Mod}$.

- $A\mathrm{Mod}$ is **closed** with respect to the above monoidal structure. The internal hom

$$\mathrm{hom} : A\mathrm{Mod}^{\mathrm{op}} \times A\mathrm{Mod} \to A\mathrm{Mod}$$

sends any two A-modules to the vector space of A-module homomorphisms between them, equipped with an A-module structure in the obvious way.

Structures and properties of $A\mathrm{Mod}$.

- $A\mathrm{Mod}$ **has duals**. The dual

$$(-)^* : A\mathrm{Mod} \to A\mathrm{Mod}^{\mathrm{op}}$$

is

$$(-)^* = \mathrm{hom}(-, A)\,.$$

The A-module V is of **finite rank** if $(V^*)^* \simeq V$.
The full subcategory of finite rank modules we denote

$$A\mathrm{Mod}_{\mathrm{fin}}\,.$$

- $A\mathrm{Mod}_{\mathrm{fin}}$ is **compact closed**, meaning that the internal hom exists and is

$$\mathrm{hom}(V, W) \simeq V^* \otimes_A W\,.$$

Example

In the context of Swan's theorem, consider modules of function algebras given by sections $\Gamma(E)$ of vector bundles $E \to X$ over some space X.

Then:

- The dual module $V^* \simeq \Gamma(E^*)$ is the space of sections of the dual bundle.
- The tensor product $V \otimes_A W$ corresponds, under to the ordinary fiberwise tensor product of vector bundles:

$$\Gamma(E) \otimes_{C(X)} \Gamma(E') \simeq \Gamma(E \otimes_{\mathrm{VectBun}} E') \,.$$

Chain complexes as internal ω-categories.

One combinatorial model for higher dimensional homotopies are ω-categories (strict, globular or cubical). The nerve of an ω-category internal to $A\mathrm{Mod}$ is a simplicial A-module.
The famous Dold-Kan correspondence says that by forgetting lots of face maps except one, and restricting it to the kernel of some of the other face maps, one obtains from a simplicial A-module a non-negatively graded chain complex of A-modules without loosing information.

Dold-Kan correspondence

Forming the normalized chain complex from a simplicial A-module is an equivalence of categories

$$A\mathrm{Mod}^{\Delta^{\mathrm{op}}} \xrightarrow{\simeq} \mathrm{Ch}_\bullet^+(A\mathrm{Mod}) \ .$$

This equivalence is just the first in a longer list.

Brown and Higgins [?]

Let $\mathcal{A}$ be an abelian category. Then the following categories, internal to $\mathcal{A}$, are all equivalent:

- simplicial objects
- chain complexes
- crossed complexes
- cubical sets with connections
- cubical ω-groupoids with connections
- globular ω-groupoids.

Remark

There are one or two sign conventions that need to be fixed once and for all before dealing with complexes. With an eye towards maximal harmony with applications to the BV complex, we shall adopt the following convention

- All our complexes will be *co*chain complexes, meaning that the differentials *increase* the degree by one, with in general no restriction on the sign of the degree.
- Ordinary chain complexes are then recovered as cochain complexes of non-*positive* degree.

degree	$+$	0	-
interpretation		ordinary vector space	
		$\underbrace{\qquad\qquad\qquad\qquad}_{\omega-\text{vector space}}$	

Cochain complexes of A-modules

Definition

We denote by $\mathbf{Ch}^\bullet(A)$ the category of A cochain complexes in $A\mathrm{Mod}$.

Objects V are cochain complexes of A-modules

$$V^\bullet = (\cdots \longrightarrow V^{-2} \xrightarrow{d_V^{-1}} V^{-1} \xrightarrow{d_V^0} V^0 \xrightarrow{d_V^1} V^1 \xrightarrow{d_V^2} V^2 \longrightarrow \cdots),$$

$$d_V^{k+1} \circ d_V^k = 0 \quad \forall k \in \mathbb{Z}.$$

Morphisms $f^\bullet : V^\bullet \to W^\bullet$ are cochain maps

$$\begin{tikzcd}
\cdots \arrow[r] & V^{-2} \arrow[r, "d_V^{-1}"] \arrow[d, "f^{-2}"] & V^{-1} \arrow[r, "d_V^0"] \arrow[d, "f^{-1}"] & V^0 \arrow[r, "d_V^1"] \arrow[d, "f^0"] & V^1 \arrow[r, "d_V^2"] \arrow[d, "f^1"] & V^2 \arrow[r] \arrow[d, "f^1"] & \cdots \\
\cdots \arrow[r] & W^{-2} \arrow[r, "d_W^{-1}"] & W^{-1} \arrow[r, "d_W^0"] & W^0 \arrow[r, "d_W^1"] & W^1 \arrow[r, "d_W^2"] & W^2 \arrow[r] & \cdots
\end{tikzcd} .$$

We assume all chain complexes to be nontrivial only in *finitely* many degrees.
It is useful to distinguish the full subcategories

- $\mathbf{Ch}^-(A)$ of cochain complexes concentrated in *non-positive* degree;
- $\mathbf{Ch}^+(A)$ of cochain complexes concentrated in *non-negative* degree.

One may think of $\mathbf{Ch}^-(A)$ as ω-vector spaces and of $\mathbf{Ch}^+(A)$ as ω-co-vector spaces.
Using the notation

$$V_n := V^{-n}$$

we can neatly switch back and forth between the two pictures.

Remark

- If we forget the differentials, i.e. if we look at cochain complexes with all differentials trivial (the 0-maps), then these are the same as $\mathbb{Z}$-graded A-modules.
- When we have nontrivial differentials, their nilpotency, $d^2 = 0$, necessarily imposes, as discussed below, on these graded vector spaces the structure of *super*vector spaces: the symmetric braiding is the nontrivial $\mathbb{Z}_2$-grading that introduces a sign whenever two odd-graded components are interchanged.

Structure and properties and of $\mathbf{Ch}^\bullet(A)$.

We list some useful facts about $\mathbf{Ch}^\bullet(A)$.

$\mathbf{Ch}^\bullet(A)$ is **symmetric monoidal** with the tensor product

$$\otimes : \mathbf{Ch}^\bullet(A) \times \mathbf{Ch}^\bullet(A) \to \mathbf{Ch}^\bullet(A)$$

defined by

$$V^\bullet \otimes W^\bullet = \left(\cdots \longrightarrow (V^\bullet \otimes W^\bullet)^n \xrightarrow{\quad d^{n+1}_{V^\bullet \otimes W^\bullet} \quad} (V^\bullet \otimes W^\bullet)^{n+1} \longrightarrow \cdots \right) .$$

$$\left(\cdots \to \left(\bigoplus_{k \in \mathbb{Z}} V^k \otimes_A W^{n-k} \right) \xrightarrow{\bigoplus_{k \in \mathbb{Z}} \left(d_V^{k+1} \otimes_A \mathrm{Id}_{W^{n-k}} + (-1)^k \mathrm{Id}_{V^k} \otimes_A d_W^{n-k+1} \right)} \left(\bigoplus_{k \in \mathbb{Z}} V^k \otimes_A W^{n-k+1} \right) \to \cdots \right)$$

Remark.

The signs appearing here are crucial. Their nature is fixed entirely by the requirement that the tensor product is again a chain complex, i.e. by the requirement that $(d_{V\otimes W})^2 = 0$. As we will see in the following, this will also imply that our modules are subject to the nontrivial symmetric braiding which introduces a sign whenever two odd-graded modules are interchanged. All this follows just from the nilpotency condition $d^2 = 0$.
One way to understand the precise nature of the signs above is to note that when forming the tensor product $V \otimes W$, we obtain the double complex

$$
\begin{tikzcd}
 & \vdots \arrow[d, "d_W^m"'] & \vdots \arrow[d, "d_W^m"'] & \\
\cdots \arrow[r, "d_V^n"] & V^n \otimes W^m \arrow[r, "d_V^{n+1}"] \arrow[d, "d_W^{m+1}"] & V^{n+1} \otimes W^m \arrow[r] \arrow[d, "d_W^{m+1}"] & \cdots \\
\cdots \arrow[r, "d_V^n"] & V^n \otimes W^{m+1} \arrow[r, "d_V^{n+1}"] \arrow[d] & V^{n+1} \otimes W^{m+1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}
$$

as an intermediate step. The squares commute, meaning that d_V and d_W commute.

So we form

$$\tilde{d}_W := (-1)^{\deg_V} d_W$$

and then the nilpotent differential

$$d_{V \oplus W} := d_V + \tilde{d}_W \,.$$

The **tensor unit** is

$$I^\bullet := \left(\cdots \xrightarrow{d_I^{-1}} 0 \xrightarrow{d_I^{0}} A \xrightarrow{d_I^{1}} 0 \xrightarrow{d_I^{2}} \cdots \right)$$

$$\cdots \xrightarrow{0} 0 \xrightarrow{0} A \xrightarrow{0} 0 \longrightarrow \cdots$$

The **symmetric braiding**

$$\begin{tikzcd}[column sep=small] \mathbf{Ch}^\bullet(A) \times \mathbf{Ch}^\bullet(A) \arrow[rr, "\otimes"] \arrow[dr, "\sigma"'] & \arrow[d, phantom, "\Downarrow b"] & \mathbf{Ch}^\bullet(A) \otimes \mathbf{Ch}^\bullet(A) \\ & \mathbf{Ch}^\bullet(A) \times \mathbf{Ch}^\bullet(A) \arrow[ur, "\otimes"'] & \end{tikzcd}$$

with

$$\sigma : \mathbf{Ch}^\bullet(A) \times \mathbf{Ch}^\bullet(A) \to \mathbf{Ch}^\bullet(A) \times \mathbf{Ch}^\bullet(A)$$

the exchange of factors is

$$b^n_{V^\bullet,W^\bullet} : (\bigoplus_k V^k \otimes_A W^{n-k}) \xrightarrow{\bigoplus_k (-1)^{k(n-k)}} (\bigoplus_k W^{n-k} \otimes_A V^k) \,.$$

The signs here ensure the required naturality

$$\begin{tikzcd}
(\bigoplus_k V^k \otimes_A W^{n-k}) \arrow[r, "\bigoplus_k (-1)^{k(n-k)}"] \arrow[d, "d_V \otimes_A \mathrm{Id} + (-1)^k \mathrm{Id} \otimes_A d_W"'] & (\bigoplus_k W^{n-k} \otimes_A V^k) \arrow[d, "d_W \otimes_A \mathrm{Id} + (-1)^{n-k} \mathrm{Id} \otimes"] \\
(\bigoplus_k V^k \otimes_A W^{n-k+1}) \arrow[r, "\bigoplus_k (-1)^{k(n-k+1)}"] & (\bigoplus_k W^{n-k+1} \otimes_A V^k)
\end{tikzcd}$$

Ch$^\bullet(A)$ is **enriched over** $A\mathrm{Mod}$

$$\mathrm{Hom} : \mathrm{Ch}^\bullet(A)^{\mathrm{op}} \times \mathrm{Ch}^\bullet(A)^{\mathrm{op}} \to A\mathrm{Mod}$$

$\mathbf{Ch}^\bullet(A)$ is closed. So for all $V \in \mathbf{Ch}^\bullet(A)$ there is an internal hom functor

$$\mathrm{hom}(V, -) : \mathbf{Ch}^\bullet(A) \to \mathbf{Ch}^\bullet(A)$$

right adjoint to

$$- \otimes V : \mathbf{Ch}^\bullet(A) \to \mathbf{Ch}^\bullet(A)$$

meaning that

$$\mathrm{Hom}(U \otimes V, W) \simeq \mathrm{Hom}(U, \mathrm{hom}(V, W))$$

naturally in U and W.

This internal hom-complex $\mathrm{hom}(V,W)$ looks as follows:

$$\mathrm{hom}(V,W) :=$$

$$\left(\cdots \longrightarrow \mathrm{hom}(V,W)^n \xrightarrow{d^{n+1}_{\mathrm{hom}(V,W)}} \mathrm{hom}(V,W)^{n+1} \longrightarrow \cdots\right)$$

$$= \left(\cdots \to \bigoplus_k \mathrm{hom}_{A\mathrm{Mod}}(V^k, W^{k+n}) \xrightarrow{\bigoplus\left((d_W^{k+n+1}\circ -) - (-1)^n(-\circ d_V^k)\right)} \bigoplus_k \mathrm{hom}_{A\mathrm{Mod}}(V^k, W^{k+n+1}) \to \cdots\right)$$

The differential $d_{\mathrm{hom}(V,W)}$ here can be understood from looking at the evaluation map

$$\mathrm{ev} : \mathrm{hom}(V, W) \otimes V \to W$$

which exists by general nonsense on internal homs. Let $f \in \oplus_k \mathrm{Hom}(V^k, W^{k+n})$ be any homogeneous element in the internal hom and $x \in V^m$. Write $f(x)$ for $\mathrm{ev}(f, x)$. Then the fact that ev is a cochain morphism says that

$$d_W(f(v)) = (d_{\mathrm{hom}(V,W)} f)(x) + (-1)^n f(d_V x) \,.$$

Solving this for $d_{\mathrm{hom}(V,W)} f$ yields the action of the differential as given above.

Remark

Notice that it is the space of *cocycles* in degree 0 of $\mathrm{hom}(V, W)$ that corresponds to the external $\mathrm{Hom}(V, W)$:

$$Z^0(\mathrm{hom}(V, W)) \simeq \mathrm{Hom}(V, W) = \ker((d_W \circ -) - (- \circ d_V)) .$$

$\mathbf{Ch}^\bullet(A)$ has duals. Since we have a tensor unit I and an internal hom, we have duality

$$(\cdot)^* : \mathbf{Ch}^\bullet(A) \to \mathbf{Ch}^\bullet(A)^{\mathrm{op}}$$

given by

$$V^* := \mathrm{hom}(V, I)\,.$$

Using the above one finds

$$V^* = \left(\cdots \longrightarrow (V^*)^n \xrightarrow{d_{V^*}^{n+1}} (V^*)^{n+1} \longrightarrow \cdots \right) .$$

$$= \left(\cdots \longrightarrow (V^{-n})^* \xrightarrow{-(-1)^n (d_V^{-n})^*} (V^{-n-1})^* \longrightarrow \cdots \right)$$

The unit $i : I \to V \otimes V^*$ is

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & A \arrow[r] \arrow[d, "\bigoplus_k i_{V^k}"] & 0 \arrow[r] \arrow[d] & \cdots , \\
\cdots \arrow[r] & \bigoplus_k V^k \otimes_A (V^{k+1})^* \arrow[r] & \bigoplus_k V^k \otimes_A (V^k)^* \arrow[r] & \bigoplus_k V^k \otimes_A (V^{k-1})^* \arrow[r] & \cdots
\end{tikzcd}$$

while the counit $e : V^* \otimes V \to I$ is

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_k V^k \otimes_A (V^{k+1})^* \arrow[r] \arrow[d, "0"] & \bigoplus_k V^k \otimes_A (V^k)^* \arrow[r] \arrow[d, "\bigoplus_k e_{V^k}"] & \bigoplus_k V^k \otimes_A (V^{k-1})^* \arrow[r] \arrow[d, "0"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}.$$

Let $\mathbf{Ch}^\bullet_{\mathrm{fin}}(A)$ be the full subcategory on those chain complexes that are bounded (only finitely many entries are nonvanishing) and all whose entries satisfy $\mathrm{hom}_{\mathrm{AMod}}(V^k, W^l) \simeq (V^k)^* \otimes_A W^l$.
Then: $\mathbf{Ch}^\bullet_{\mathrm{fin}}(A)$ is **compact closed** meaning that the internal hom is isomorphic to a tensor product

$$\mathrm{hom}(V, W) \simeq W \otimes V^* .$$

$\mathbf{Ch}^\bullet(A)$ has plenty of other nice structures. In particular, it naturally is a **model category**.

dg-Algebras and dg-coalgebras

The crucial (but simple) fact underlying most of what we are doing here is:

Observation

Monoids in $\mathbf{Ch}^\bullet(A)$, i.e. cochain complexes V equipped with a product morphism $\mu : V \otimes V \to V$ and a unit morphism $i : I \to V$ such that μ is associative

$$\begin{tikzcd} V \otimes V \otimes V \arrow[r, "\mu\otimes \mathrm{Id}_V"] \arrow[d, "\mathrm{Id}_V\otimes\mu"'] & V \otimes V \arrow[d, "\mu"] \\ V \otimes V \arrow[r, "\mu"] & V \end{tikzcd}$$

and unital

$$\begin{tikzcd} I \otimes V \arrow[rr] \arrow[dr, "i\otimes \mathrm{Id}_V"'] & & V \\ & V \otimes V \arrow[ur, "\mu"'] & \end{tikzcd}$$

are precisely **differential graded algebras (dg-algebras)**.

dg-algebra

A dg-algebra is an associative graded algebra $(V, \cdot)$ equipped with a graded derivation

$$d : V \to V$$

of degree $+1$ that squares to 0,

$$d^2 = 0 \,.$$

Of ocurse this has a co-version:

Defintion

Comonoids in $\mathbf{Ch}^\bullet(A)$, i.e. cochain complexes V equipped with a coproduct morphism $\delta : V \to V \otimes V$ and a counit morphism $e : V \to I$ such that δ is coassociative

$$\begin{tikzcd}
V \otimes V \otimes V & & V \otimes V \arrow[ll, "\delta\otimes\mathrm{Id}_V"'] \\
V \otimes V \arrow[u, "\mathrm{Id}_V\otimes\delta"] & & V \arrow[ll, "\delta"] \arrow[u, "\delta"']
\end{tikzcd}$$

and counital

$$\begin{tikzcd}
I \otimes V & & V \arrow[ll] \arrow[dl, "\mu"] \\
& V \otimes V \arrow[ul, "i\otimes\mathrm{Id}_V"] &
\end{tikzcd}$$

are precisely **codifferential graded coalgebras (cdg-coalgebras)**.

cdg-coalgebra

A cdg-coalgebra is a coassociative graded coalgebra $(V, \cdot)$ equipped with a graded coderivation

$$D : V \to V$$

of degree $+1$ that squares to 0,

$$D^2 = 0 \,.$$

Definition

We write

$$\mathrm{Monoids}(\mathrm{Ch}^\bullet(A))$$

for the category of monoids internal to $\mathrm{Ch}^\bullet(A)$ and

$$\mathrm{CoMonoids}(\mathrm{Ch}^\bullet(A))$$

for the category of comonoids internal to $\mathrm{Ch}^\bullet(A)$.
We write

$$\mathrm{ComMonoids}(\mathrm{Ch}^\bullet(A))$$

for the category of commutative monoids internal to $\mathrm{Ch}^\bullet(A)$ and

$$\mathrm{CoComMonoids}(\mathrm{Ch}^\bullet(A))$$

for the category of cocommutative comonoids internal to $\mathrm{Ch}^\bullet(A)$.

Quasi-free dg-(co)-algebra

We shall mainly be interested in dg-algebras that are free in a certain sense. These come from symmetric tensor powers.

Definition

The **symmetric tensor prodct** of an object V in $\mathrm{Ch}^\bullet(A)$ with itself is

$$
\begin{aligned}
V \wedge V &:= \ker(\mathrm{Id}_{V\otimes V} - b_{V,V}) \\
&= \mathrm{im}(\frac{1}{2}(\mathrm{Id}_{V\otimes V} + b_{V,V}))\,,
\end{aligned}
$$

where $b_{V,W}$ is the component of the symmetric braiding, described above.
Similarly the nth symmetric tensor power

$$\wedge^n V$$

is defined by symmetrizing, using $b_{V,V}$, over all $n!$ permutations.

Remark.

- Notice that for chain complexes concentrated in degree 0, the symmetric tensor product coincides with the usual symmetric tensor product of plain A-modules. For chain complexes with all differentials vanishing it corresponds to the graded symmetric product of the corresponding graded A-modules.
- The definition of $V \wedge V$ in terms of the image of the projector

$$\mathrm{sym} := \frac{1}{2}(\mathrm{Id}_{V \otimes V} + b_{V,V})$$

is convenient (see below), but does need to assume that we are working over a field not of characteristic 2.

Observation

A graded-commutative dg-algebra is a monoid (V, μ) in $\mathbf{Ch}^\bullet(A)$ whose product factors through $V \wedge V$.

$$\begin{tikzcd} V \otimes V \arrow[rr, "\mu"] \arrow[d, "\frac{1}{2}(\mathrm{Id}_{V\otimes V} + b_{V,V})"'] & & V \\ V \wedge V \arrow[r, hook] & V \otimes V \arrow[ur, "\mu"'] & \end{tikzcd}$$

Definition

The **tensor algebra** over a complex V is the complex

$$TV := \bigoplus_{n\in\mathbb{N}} \underbrace{V\otimes\cdots\otimes V}_{n} := I \oplus V \oplus (V\otimes V) \oplus \cdots .$$

equipped with the tautological monoid structure
The **symmetric tensor algebra** over a complex V is the complex

$$\Lambda^\bullet V := \bigoplus_{n\in\mathbb{N}} \wedge^n V = I \oplus V \oplus (V\wedge V) \oplus \cdots .$$

The monoid structure $\cdot : \Lambda V \otimes \Lambda V \to \Lambda V$ on this is the one from above, composed with the graded symmetrization projector

$$\mu : \ V^{\wedge k} \otimes V^{\wedge l} \xrightarrow{\mathrm{sym}} V^{\wedge(k+l)} \ .$$

Here the infinite sum is defined to be the direct limit

$$\bigoplus_{n\in\mathbb{V}} \wedge^n V := \lim_{\rightarrow} \left(\bigoplus_{n=0}^{k} \wedge^n V \right) .$$

Example

Let V be an ordinary vector space, regarded as a chain complex concentrated in degree 0, with $A = k$ the ground field. Then

$$TV$$

is the ordinary tensor algebra over V,

$$\wedge V$$

is the free symmetric tensor algebra (the **bosonic Fock space over** V) and

$$\wedge(V[1])$$

is the (free graded-commutative) **Grassmann algebra** over V (**the fermionic Fock space over** V).

Example

Let $(\mathfrak{g}, [\cdot,\cdot])$ a finite dimensional Lie algebra over our ground field. Then the **Chevalley-Eilenberg algebra** $CE(\mathfrak{g})$ of that Lie algebra is the graded commutative dg-algebra obtained by equipping

$$\Lambda(\mathfrak{g}^*[1])$$

with the differential

$$d : \Lambda(\mathfrak{g}^*[1]) \to (\Lambda(\mathfrak{g}^*[1]) \wedge \Lambda(\mathfrak{g}^*[1]))[-1]$$

defined by

$$d|_{\mathfrak{g}^*[1]} := [\cdot,\cdot]^* \,.$$

The cohomology of the corresponding complex is, by definition, the Lie algebra cohomology of $\mathfrak{g}$ (with values in the trivial module).

Example

Let $(\mathfrak{g}, [\cdot,\cdot])$ a finite dimensional Lie algebra over our ground field. Then the **Weil algebra** $W(\mathfrak{g})$ of that Lie algebra is the graded commutative dg-algebra obtained by equipping

$$\Lambda(\mathfrak{g}^*[1] \oplus \mathfrak{g}^*[2])$$

with the differential

$$d : \Lambda(\mathfrak{g}^*[1]) \to (\Lambda(\mathfrak{g}^*[1]) \wedge \Lambda(\mathfrak{g}^*[1]))[-1]$$

defined by

$$d|_{\mathfrak{g}^*[1]} := [\cdot,\cdot]^* + s^*$$

and

$$d(s^*(x)) := s^* dx$$

for all $x \in \mathfrak{g}^*[1]$ and with $s : \mathfrak{g}[2] \to \mathfrak{g}[1]$ the canonical isomorphism. The closed elements in $\Lambda\mathfrak{g}^*[2] \subset \Lambda(\mathfrak{g}^*[1] \oplus \mathfrak{g}^*[2])$ are the symmetric invariant polynomials on $\mathfrak{g}$.

Remark

In order to put this into perspective, I make the following remark, without, at this point, trying to actually describe or explain any of these statements.

The Weil algebra $W(\mathfrak{g})$ arises from $CE(\mathfrak{g})$ in a universal way. All of the following are synonymous:

- $W(\mathfrak{g})$ is the **mapping cone** of the identity map on $CE(\mathfrak{g})$. .
- $W(\mathfrak{g})$ is the **homotopy quotient** of the identity map on $CE(\mathfrak{g})$.
- $W(\mathfrak{g})$ is the **weak cokernel** of the identity map on $CE(\mathfrak{g})$.

Moreover

- $CE(\mathfrak{g})$ plays the role of differential forms on G.
- $W(\mathfrak{g})$ plays the role of differential forms on EG.
- $\mathrm{inv}(\mathfrak{g})$, the graded commutative algebra of closed elements in $W(\mathfrak{g})|_{\wedge \mathfrak{g}^*[2]}$, plays the role of differential forms on BG.

And we have a canonical sequence

$$\begin{tikzcd} G \arrow[r, hook] & EG \arrow[r, two heads] & BG \end{tikzcd}$$

$$\begin{tikzcd} CE(\mathfrak{g}) & W(\mathfrak{g}) \arrow[l, two heads] & \mathrm{inv}(\mathfrak{g}) \arrow[l, hook'] \end{tikzcd}$$

The following fact will be of importance:

Observation

$$\Lambda V = \wedge^{\infty}(I \oplus V)$$

In particular

Observation

If $V \in \mathrm{bf}\, Ch^{\bullet}(A)$ contains in degree 0 just the tensor unit

$$V^0 = A$$

then

$$\wedge^{\infty} V$$

naturally is a monoid in $\mathbf{Ch}^{\bullet}(A)$.

previous: ∞-vector spaces

Lie ∞-algebras: differential graded (co)algebra

next: ∞-configuration spaces

Given a graded vector space V, the *tensor space* $T^\bullet(V) := \bigoplus_{n=0} V^{\otimes n}$ with V^0 being the ground field. We will denote by $T^a(V)$ the *tensor algebra* with the concatenation product on $T^\bullet(V)$:

$$x_1 \otimes x_2 \otimes \cdots \otimes x_p \bigotimes x_{p+1} \otimes \cdots \otimes x_n \mapsto x_1 \otimes x_2 \otimes \cdots \otimes x_n$$

and by $T^c(V)$ the *tensor coalgebra* with the deconcatenation product on $T^\bullet(V)$:

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto \sum_{p+q=n} x_1 \otimes x_2 \otimes \cdots \otimes x_p \bigotimes x_{p+1} \otimes \cdots \otimes x_n .$$

The *graded symmetric algebra* $\wedge^\bullet(V)$ is the quotient of the tensor algebra $T^a(V)$ by the graded action of the symmetric groups $\mathbf{S}_n$ on the components $V^{\otimes n}$. The *graded symmetric coalgebra* $\vee^\bullet(V)$ is the sub-coalgebra of the tensor coalgebra $T^c(V)$ fixed by the graded action of the symmetric groups $\mathbf{S}_n$ on the components $V^{\otimes n}$.

Remark

$\vee^\bullet(V)$ is spanned by graded symmetric tensors

$$x_1 \vee x_2 \vee \cdots \vee x_p$$

for $x_i \in V$ and $p \geq 0$, where we use $\vee$ rather than $\wedge$ to emphasize the coalgebra aspect, e.g.

$$x \vee y = x \otimes y \pm y \otimes x .$$

In characteristic zero, the graded symmetric algebra can be identified with a sub-algebra of $T^a(V)$ but that is unnatural and we will try to avoid doing so. The coproduct on $\vee^\bullet(V)$ is given by

$$\Delta(x_1 \vee x_2 \cdots \vee x_n) = \sum_{p+q=n} \sum_{\sigma \in \mathrm{Sh}(p,q)} \epsilon(\sigma)(x_{\sigma(1)} \vee x_{\sigma(2)} \cdots x_{\sigma(p)}) \otimes (x_{\sigma(p+1)} \vee \cdot$$

Here

- $\mathrm{Sh}(p,q)$ is the subset of all those bijections (the "unshuffles") of $\{1,2,\cdots,p+q\}$ that have the property that $\sigma(i) < \sigma(i+1)$ whenever $i \neq p$;
- $\epsilon(\sigma)$, which is shorthand for $\epsilon(\sigma, x_1 \vee x_2, \cdots x_{p+q})$, the Koszul sign, defined by

$$x_1 \vee \cdots \vee x_n = \epsilon(\sigma) x_{\sigma(1)} \vee \cdots x_{\sigma(n)} \,.$$

Definition (L_∞-algebra)

An L_∞-algebra $\mathfrak{g} = (\mathfrak{g}, D)$ is a $\mathbb{N}_+$-graded vector space $\mathfrak{g}$ equipped with a degree -1 differential coderivation

$$D : \vee^\bullet \mathfrak{g} \to \vee^\bullet \mathfrak{g}$$

on the graded co-commutative coalgebra generated by $\mathfrak{g}$, such that $D^2 = 0$. This induces a differential

$$d_{\mathrm{CE}(\mathfrak{g})} : \mathrm{Sym}^\bullet(\mathfrak{g}) \to \mathrm{Sym}^{\bullet+1}(\mathfrak{g})$$

on graded-symmetric multilinear functions on $\mathfrak{g}$. When $\mathfrak{g}$ is finite dimensional this yields a degree $+1$ differential

$$d_{\mathrm{CE}(\mathfrak{g})} : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^*$$

on the graded-commutative algebra generated from $\mathfrak{g}^*$. This is the Chevalley-Eilenberg dg-algebra corresponding to the L_∞-algebra $\mathfrak{g}$.

Remark

That the original definition of L_∞-algebras in terms of multibrackets yields a codifferential coalgebra as above was shown in [Lada-Stasheff]. That every such codifferential comes from a collection of multibrackets this way is due to [Lada-Markl].

Example

For $(\mathfrak{g}[-1], [\cdot,\cdot])$ an ordinary Lie algebra (meaning that we regard the vector space $\mathfrak{g}$ to be in degree 1), the corresponding Chevalley-Eilenberg qDGCA is

$$\mathrm{CE}(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}^*, d_{\mathrm{CE}(\mathfrak{g})})$$

with

$$d_{\mathrm{CE}(\mathfrak{g})} : \mathfrak{g}^* \xrightarrow{[\cdot,\cdot]^*} \mathfrak{g}^* \wedge \mathfrak{g}^* \ .$$

Example

If we let $\{t_a\}$ be a basis of $\mathfrak{g}$ and $\{C^a{}_{bc}\}$ the corresponding structure constants of the Lie bracket $[\cdot,\cdot]$, and if we denote by $\{t^a\}$ the corresponding basis of $\mathfrak{g}^*$, then we get

$$d_{\mathrm{CE}(\mathfrak{g})} t^a = -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c \,.$$

If $\mathfrak{g}$ is concentrated in degree $1, \ldots, n$, we also say that $\mathfrak{g}$ is a **Lie** *n*-**algebra**.
Notice that built in we have a shift of degree for convenience, which makes ordinary Lie 1-algebras be in degree 1 already. In much of the literature a Lie n-algebra would be based on a vector space concentratred in degrees 0 to $n-1$. An ordinary Lie algebra is a Lie 1-algebra. Here the coderivation differential $D = [\cdot, \cdot]$ is just the Lie bracket, extended as a coderivation to $\vee^\bullet \mathfrak{g}$, with $\mathfrak{g}$ regarded as being in degree 1.

In the rest of the discussion we assume, just for simplicity and since it is sufficient for our applications, all $\mathfrak{g}$ to be finite-dimensional. Then, by the above, these L_∞-algebras are equivalently conceived of in terms of their dual Chevalley-Eilenberg algebras, $\mathrm{CE}(\mathfrak{g})$, as indeed every quasi-free differential graded commutative algebra ("qDGCA", meaning that it is free as a graded commutative algebra) corresponds to an L_∞-algebra. We will find it convenient to work entirely in terms of qDGCAs, which we will usually denote as $\mathrm{CE}(\mathfrak{g})$.

While not interesting in themselves, truly free differential algebras are a useful tool for handling quasi-free differential algebras.

Definition

We say a qDGCA is *free* (even as a differential algebra) if it is of the form

$$\mathrm{F}(V) := (\wedge^\bullet(V^* \oplus V^*[1]), d_{\mathrm{F}(V)})$$

with

$$d_{\mathrm{F}(V)}|_{V^*} = \sigma : V^* \to V^*[1]$$

the canonical isomorphism and

$$d_{\mathrm{F}(V)}|_{V^*[1]} = 0\,.$$

Remark

Such algebras are indeed free in that they satisfy the universal property: given any linear map $V \to W$, it uniquely extends to a morphism of qDGCAs $F(V) \to (\bigwedge^\bullet(W^*), d)$ for any choice of differential d.

Example

The free qDGCA on a 1-dimensional vector space in degree 0 is the graded commutative algebra freely generated by two generators, t of degree 0 and dt of degree 1, with the differential acting as $d : t \mapsto dt$ and $d : dt \mapsto 0$. In rational homotopy theory, this models the interval $I = [0,1]$. The fact that the qDGCA is free corresponds to the fact that the interval is homotopy equivalent to the point.
We will be interested in qDGCAs that arise as mapping cones of morphisms of L_∞-algebras.

"mapping cone" of qDGCAs

Let

$$\mathrm{CE}(\mathfrak{h}) \xleftarrow{t^*} \mathrm{CE}(\mathfrak{g})$$

be a morphism of qDGCAs. The mapping cone of t^*, which we write $\mathrm{CE}(\mathfrak{h} \xrightarrow{t} g)$, is the qDGCA whose underlying graded algebra is

$$\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{h}^*[1])$$

and whose differential d_{t^*} is such that it acts as

$$d_{t^*} = \begin{pmatrix} d_{\mathfrak{g}} & 0 \\ t^* & d_{\mathfrak{h}} \end{pmatrix} .$$

Definition (Weil algebra of an L_∞-algebra)

The mapping cone of the identity on $\mathrm{CE}(\mathfrak{g})$ is the Weil algebra

$$\mathrm{W}(\mathfrak{g}) := \mathrm{CE}(\mathfrak{g} \xrightarrow{\mathrm{Id}} \mathfrak{g})$$

of $\mathfrak{g}$.

Proposition

For $\mathfrak{g}$ an ordinary Lie algebra this does coincide with the ordinary Weil algebra of $\mathfrak{g}$.

The Weil algebra has two important properties.

Proposition

The Weil algebra $\mathrm{W}(\mathfrak{g})$ of any L_∞-algebra $\mathfrak{g}$

- is isomorphic to a free differential algebra $\mathrm{W}(\mathfrak{g}) \simeq \mathrm{F}(\mathfrak{g})$, and hence is contractible;
- has a canonical surjection $\mathrm{CE}(\mathfrak{g}) \overset{i^*}{\twoheadleftarrow} \mathrm{W}(\mathfrak{g})$.

As a corollary we obtain

Corollary

For $\mathfrak{g}$ any L_∞-algebra, the cohomology of $\mathrm{W}(\mathfrak{g})$ is trivial.

Remark

As we will shortly see, $\mathrm{W}(\mathfrak{g})$ plays the role of the algebra of differential forms on the universal $\mathfrak{g}$-bundle. The surjection $CE(\mathfrak{g}) \overset{i^*}{\twoheadleftarrow} W(\mathfrak{g})$ plays the role of the restriction to the differential forms on the fiber of the universal $\mathfrak{g}$-bundle.

Examples

We construct large families of examples the so-called **String-like extensions** of L_∞-algebras, based on the first two of the following examples and elements in L_∞-algebra cohomology.

1. Ordinary Weil algebras as Lie 2-algebras

What is ordinarily called the Weil algebra $\mathrm{W}(\mathfrak{g})$ of a Lie algebra $(\mathfrak{g}[-1], [\cdot,\cdot])$ can, since it is again a DGCA, also be interpreted as the Chevalley-Eilenberg algebra of a Lie 2-algebra. This Lie 2-algebra we call $\mathrm{inn}(\mathfrak{g})$. It corresponds to the Lie 2-group $\mathrm{INN}(G)$ discussed in [?]:

$$\mathrm{W}(\mathfrak{g}) = \mathrm{CE}(\mathrm{inn}(\mathfrak{g}))\,.$$

We have

$$\mathrm{W}(\mathfrak{g}) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1]), d_{\mathrm{W}(\mathfrak{g})})\,.$$

1. Ordinary Weil algebras as Lie 2-algebras

Denoting by $\sigma : \mathfrak{g}^* \to \mathfrak{g}^*[1]$ the canonical isomorphism, extended as a derivation to all of $\mathrm{W}(\mathfrak{g})$, we have

$$d_{\mathrm{W}(\mathfrak{g})} : \mathfrak{g}^* \xrightarrow{[\cdot,\cdot]^* + \sigma} \mathfrak{g}^* \wedge \mathfrak{g}^* \oplus \mathfrak{g}^*[1]$$

and

$$d_{\mathrm{W}(\mathfrak{g})} : \mathfrak{g}^*[1] \xrightarrow{-\sigma \circ d_{\mathrm{CE}(\mathfrak{g})} \circ \sigma^{-1}} \mathfrak{g}^* \otimes \mathfrak{g}^*[1] \ .$$

1. Ordinary Weil algebras as Lie 2-algebras

With $\{t^a\}$ a basis for $\mathfrak{g}^*$ as above, and $\{\sigma t^a\}$ the corresponding basis of $\mathfrak{g}^*[1]$ we find

$$d_{\mathrm{W}(\mathfrak{g})} : t^a \mapsto -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c + \sigma t^a$$

and

$$d_{\mathrm{W}(\mathfrak{g})} : \sigma t^a \mapsto -C^a{}_{bc} t^b \sigma t^c \, .$$

1. Ordinary Weil algebras as Lie 2-algebras

The Lie 2-algebra $\mathrm{inn}(\mathfrak{g})$ is, in turn, nothing but the strict Lie 2-algebra as in the third example below, which comes from the infinitesimal crossed module $(\mathfrak{g} \xrightarrow{\mathrm{Id}} \mathfrak{g} \xrightarrow{\mathrm{ad}} \mathrm{der}(\mathfrak{g}))$.

2. Shifted $\mathfrak{u}(1)$.

By the above, the qDGCA corresponding to the Lie algebra $\mathfrak{u}(1)$ is simply

$$\mathrm{CE}(\mathfrak{u}(1)) = (\wedge^\bullet \mathbb{R}[1], d_{\mathrm{CE}(\mathfrak{u}(1))} = 0)\,.$$

We write

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) = (\wedge^\bullet \mathbb{R}[n], d_{\mathrm{CE}(b^n\mathfrak{u}(1))} = 0)$$

for the Chevalley-Eilenberg algebras corresponding to the Lie n-algebras $b^{n-1}\mathfrak{u}(1)$.

3. Infinitesimal crossed modules and strict Lie 2-algebras.

An *infinitesimal crossed module* is a diagram

$$(\mathfrak{h} \xrightarrow{t} \mathfrak{g} \xrightarrow{\alpha} \mathrm{der}(\mathfrak{h}))$$

of Lie algebras where t and α satisfy two compatibility conditions. These conditions are equivalent to the nilpotency of the differential on

$$\mathrm{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) := (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{h}^*[1]), d_t)$$

defined by

$$\begin{aligned} d_t|_{\mathfrak{g}^*} &= [\cdot,\cdot]^*_{\mathfrak{g}} + t^* \\ d_t|_{\mathfrak{h}^*[1]} &= \alpha^* , \end{aligned} \qquad (1)$$

where we consider the vector spaces underlying both $\mathfrak{g}$ and $\mathfrak{h}$ to be in degree 1.

3. Infinitesimal crossed modules and strict Lie 2-algebras.

Here in the last line we regard α as a linear map $\alpha : \mathfrak{g} \otimes \mathfrak{h} \to \mathfrak{h}$. The Lie 2-algebras $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ thus defined are called strict Lie 2-algebras: these are precisely those Lie 2-algebras whose Chevalley-Eilenberg differential contains at most co-binary components.

previous: Lie ∞-algebras

∞-Configuration spaces: the BV-complex

- Differential forms on spaces of maps
- The Chevalley-Eilenberg complex.
- The Koszul-Tate complex.
- The full Batalin-Vilkovisky-complex.

next: Examples

Differential forms on spaces of maps

As mentioned in the introduction, the starting point of BV-formalism is, in our language, the idea that we replace the configuration *space* of our system with the Lie n-algebroid whose objects are the configurations, and whose morphisms are the gauge transformations.

Then we need to ask: **how do we obtain the configuration Lie** n-**algebroids**?

The answer is: **by constructing the configuration space internally**.

This means:

Observation

The configuration space is often taken to be the *set* of maps from parameter space par to target space tar

$$\mathrm{conf} = \mathrm{Hom}(\mathrm{par}, \mathrm{tar})\,.$$

Instead of considering this *external* Hom, we should form the *internal* hom-object

$$\mathrm{conf} = \mathrm{hom}(\mathrm{par}, \mathrm{tar})\,.$$

Remark: the AKSZ approach

The observation that the BV-formalism finds its natural home when the configuration space is constructed *internally* is, more or less implicitly, due to [Alexandrov–Schwarz-Zaboronsky], reviewed by [Roytenberg].
This author here, however, had his problems with extracting the abstract inner-hom construction which the component-based formulas of these two articles could correspond to.
The following construction is supposed to remedy this. Our examples gauge theory and higher gauge theory show that and how this works.

In quantum field theory, we are faced with the problem of making sense of a diagram of the form

$$\begin{tikzcd} & \mathrm{maps}(\mathrm{par}, \mathrm{tar}) \otimes \mathrm{par} \arrow[dl] \arrow[dr, "\mathrm{ev}"] & & \\ \mathrm{par} & & \mathrm{tar} \arrow[r, "\mathrm{tra}"] & \mathrm{phas} \end{tikzcd}$$

in some ambient category T.
We write

$$\mathrm{conf} := \mathrm{maps}(\mathrm{par}, \mathrm{tar})\,.$$

The morphism tra induces the **action**

$$\exp(S) : 1 \to \mathrm{maps}(\mathrm{conf}, \mathrm{maps}(\mathrm{par}, \mathrm{tar}))\,.$$

DGCAs of maps

We now explain and put into perspective the following definition, which is used extensively in our Examples:

Deifnition

For A and B any two DGCAs, we write

$$\mathrm{maps}(B, A) := \Omega^\bullet(\mathrm{Hom}_{\mathrm{DGCAs}}(B, A \otimes \Omega^\bullet(-)))$$

for the DGCA of differential forms on the presheaf over manifolds whose set of plots on any domain U is $\mathrm{Hom}_{\mathrm{DGCAs}}(B, A \otimes \Omega^\bullet(U))$.

An approximation to the internal hom in DGCAs

Throughout I write capital Hom for Hom-*sets* and lower case hom for internal Hom-objects, or their "approximations" to be discussed here.

We write $\mathrm{DGCA}s$ for the category whose objects are differential graded commutative algebras in non-negative degree, and

$$S^{\infty} := \mathrm{Set}^{S^{\mathrm{op}}}$$

for the category of generalized smooth spaces, namely of presheaves over the site S, which is any one of the sites of manifolds, the site of open subsets of $\mathbb{R} \cup \mathbb{R}^2 \cup \mathbb{R}^3 \cup \cdots$, or the like.

We have contravariant functors going back and forth between these two categories, forming an adjunction.

The functor

$$\Omega^\bullet(\cdot) : S^\infty \to \mathrm{DCGA}s$$

acts as

$$\Omega^\bullet : X \mapsto \mathrm{Hom}_{S^\infty}(X, \Omega^\bullet)\,,$$

where, in turn, here on the right $\Omega^\bullet$ denotes the smooth space of all differential forms, given by the object in S^∞ which acts as

$$\Omega^\bullet : U \mapsto \Omega^\bullet(U)\,,$$

where on the right we have the ordinary algebra of differential forms on U.

The contravariant functor

$$S^\infty \longleftarrow \mathrm{DGCAs} \;\; : \mathrm{Hom}(-, \Omega^\bullet(-))$$

acts as

$$(X_A : U \mapsto \mathrm{Hom}(A, \Omega^\bullet(U))) \longleftarrow\!\!\shortmid A \;\; : \mathrm{Hom}_{\mathrm{DGCAs}}(-, \Omega^\bullet(-))\,.$$

Notice that S^∞, being a topos, has lots of nice properties. In particular it is cartesian closed. The inner hom is

$$\mathrm{hom}_{S^\infty}(X, Y) : U \mapsto \mathrm{Hom}(X \times U, Y)\,.$$

On the other hand, the category $\mathrm{DGCA}s$ doesn't have these nice properties in general, except after one restricts to a suitably well behaved subcategory. (I suspect, though, that the above adjunction can be turned into an equivalence *on cohomology*, but I am not sure yet.)

But we can use the above adjunction to "pull back" the inernal hom in S^∞ to $\mathrm{DGCA}s$, meaning that we consider

$$\mathrm{hom}_{\mathrm{DGCA}s}(-,-):$$

$$\begin{tikzcd}
(\mathrm{DGCA}s)^{\mathrm{op}} \times \mathrm{DGCA}s \arrow[rr, "\mathrm{Hom}(-{,}\Omega^\bullet(-))^{\mathrm{op}} \times \mathrm{Hom}(-{,}\Omega^\bullet(-))^{\mathrm{op}}"] & & (S^\infty)^{\mathrm{op}} \times S^\infty \arrow[d, "\mathrm{hom}_{S^\infty}(-{,}-)"] \\
& & S^\infty \arrow[d, "\Omega^\bullet"] \\
& & \mathrm{DGCA}s
\end{tikzcd} .$$

So given DGCAs A and B, we get

$$\mathrm{hom}_{\mathrm{DGCAs}}(B, A) = \Omega^\bullet(\mathrm{hom}_{S^\infty}(X_A, X_B)) \,.$$

Proposition

We have a canonical surjection of DGCAs

$$\Omega^\bullet(\mathrm{Hom}_{\mathrm{DGCAs}}(B, A \otimes \Omega^\bullet(-))) \twoheadleftarrow \mathrm{hom}_{\mathrm{DGCAs}}(B, A) \ .$$

Proof. This comes from the canonical inclusion of smooth spaces

$$\mathrm{Hom}_{\mathrm{DGCAs}}(B, A \otimes \Omega^\bullet(-)) \hookrightarrow \mathrm{Hom}_{S^\infty}(X_A \times -, X_B)$$

which comes, on each $U \in S$, from the inclusion of sets

$$\mathrm{Hom}_{\mathrm{DGCAs}}(B, A \otimes \Omega^\bullet(U)) \hookrightarrow \mathrm{Hom}_{S^\infty}(X_A \times U, X_B)$$

which is given by

$$(f^* : B \to A \otimes \Omega^\bullet(U))$$

$$\mapsto (V \mapsto (\mathrm{Hom}_{\mathrm{DGCAs}}(A, \Omega^\bullet(V)) \times \mathrm{Hom}_S(V, U) \stackrel{\circ f^*}{\to} \mathrm{Hom}_{\mathrm{DGCAs}}(B, \Omega^\bullet(V$$

Definition: currents

For A any DGCA, we say that a current on A is a smooth linear map

$$c : A \to \mathbb{R} \,.$$

For $A = \Omega^\bullet(X)$ this reduces to the ordinary notion of currents.

Proposition

For each element $b \in B$ and current c on A, we get an element in $\Omega^\bullet(\mathrm{Hom}_{\mathrm{DGCA}s}(B, A \otimes \Omega^\bullet(-)))$ by mapping, for each $U \in S$

$$\mathrm{Hom}_{\mathrm{DGCA}s}(B, A \otimes \Omega^\bullet(U)) \to \Omega^\bullet(U)$$

$$f^* \mapsto c(f^*(b))\,.$$

If b is in degree n and c in degree $m \leq n$, then this differential form is in degree $n - m$.

The Chevalley-Eilenberg complex

The Koszul-Tate complex

Definition (dual Koszul complex)

For

$$f : M \to A$$

a morphism of A-modules with M of rank n, let

$$V := \;(0 \longrightarrow V^{-1} \xrightarrow{d_V^0} V^0 \longrightarrow 0)$$

$$(0 \longrightarrow M \xrightarrow{f} A \longrightarrow 0)$$

in $\mathbf{Ch}^\bullet(A)$ be the corresponding cochain complex in degree -1 and 0. Then

$$K[f] := \wedge^n V$$

is the (dual) Koszul complex defined by f.

Example

The crucial example of interest in the BV context is this: Let $A = C(X)$ for X some manifold. Let TX be the tangent bundle over X and $M = \Gamma(TX)$ its space of sections. Let $S \in C(X)$ be any function on X and set $f = dS(\cdot)$.

$$V := \left(0 \longrightarrow V^{-1} \xrightarrow{d_V^0} V^0 \longrightarrow 0\right) .$$

$$\left(0 \longrightarrow \Gamma(TX) \xrightarrow{dS(\cdot)} C(X) \longrightarrow 0\right)$$

In the special case that M happens to be a free A-module $M = A^n$ (e.g. sections of a trivial vector bundle in the above example), any tuple $(E_a \in A)_{a=1}^n$ of elements in A provides a morphism

$$E : A^n \to A$$

by matrix multiplication

$$E : \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix} \mapsto [E_1, \cdots, E_n] \cdot \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix} = \sum_{k=1}^{n} a^k E_k$$

we can give the following equivalent definition of the (dual) Koszul complex.

Definition

For $E_1, \cdots, E_n \in A$ a sequence of elements of A, let

$$K[E_a] := \left(0 \longrightarrow K[E_a]^{-1} \xrightarrow{d^0_{K[E_a]}} K[E_a]^0 \longrightarrow 0\right) \;.$$

$$\left(0 \longrightarrow A \xrightarrow{\cdot E_a} A \longrightarrow 0\right)$$

The complex

$$K[E_1, \cdots, E_n] := K[E_1] \otimes \cdots \otimes K[E_n]$$

is the (dual) **Koszul complex** associated with the (E_i).

Example

For two elements $K[E_1, E_2]$ looks as follows.

$$K[E_1, E_2] = K[E_1] \otimes K[E_2] := \left(0 \longrightarrow A \xrightarrow{\begin{bmatrix} -E_2 \\ E_1 \end{bmatrix}} A^2 \xrightarrow{[E_1, E_2]} A \longrightarrow 0\right) .$$

Notice that the left A factor arises as $A \otimes_A A$.

Remarks

- $K[E_1, \cdots, E_n] := \left(\cdots \longrightarrow A^{\oplus n} \xrightarrow{\sum_{a=1}^n \cdot E_k} A \longrightarrow 0 \right)$
- The cohomology of the Koszul complex in degree 0 is the quotient
  $$A/(E_1, \cdots, E_n)A, .$$
  Therefore, in the case that all other cohomology groups of the Koszul complex vanish, it provides a **resolution** of this quotient. Since all A-modules appearing in the Koszul complex are free, this resolution is necessarily *projective*.
- Therefore the cohomologies of the Koszul complex in non-vanishing degree measure the *interdependency* of the E_a. In particular, cohomology in degree -1 contains the *relations* among the E_a, namely tuples of elements $v \in A^{\oplus n}$ such that $v^a E_a = 0$ modulo the *trivial relations*.

The idea that the Koszul complex measures the independency of the elements $(E_1, \cdots, E_n)$ is made precise by the following standard definition and fact.

Definition: Regular sequence

An element $E \in A$ is called **regular** if it is not a zero-divisor and if $A/EA \neq 0$. For M an A-module, E is called M-**regular** if it is not a zero-divisor and M/EM is nonzero.
A sequence of elements $(E_1, \cdots, E_n \in A)$ is M-regular if E_k is $M/(E_1, \cdots, E_{k-1})M$-regular.

Example

Let $A = C^\infty(X)$ for some manifold X. Then a function $E \in A$ is regular, if there is no open set on which it vanishes. Let $E \to X$ be a vector bundle and $M = \Gamma(E)$ be its module of sections. Then, again, E is M-regular if it is just regular.

Fact

For A a local ring and M a finitely generated module, any sequence $(E_1, \cdots, E_n \in A)$ is M-regular precisely if the Koszul complex $K[E_1, \cdots, E_n]$ is a resolution of $M/(E_1, \cdots, E_n)$.

If the Koszul complex $K[E_1, \cdots, E_n]$ is not a resolution, we can read off from its failure of acyclicity the maximal regular sequence inside $(E_1, \cdots, E_n)$.

Fact

If precisely the highest r cohomology groups of $K[E_1, \cdots, E_2]$ vanish, then the maximal regular sequence inside $(E_1, \cdots, E_2)$ has length r.
More precisely, if

$$H^{-n+j}(K[E_1, \cdots, E_n]) = 0$$

for all $j < r$, while

$$H^{-n+r}(K[E_1, \cdots, E_n]) \neq 0$$

then every maximal regular sequence inside $(E_1, \cdots, E_n)$ has length r.

Remark

The local rings of relevance in the BV context are formal power series $K[[X_1, \cdots, X_n]]$.

Example

Let $A = K[[x_1, x_2]]$ be a power series in two variables. Then (x_1) is a regular sequence and $A/(x_1)A \simeq K[[x_2]]$. Compare this with the example **??** below.

Example.

Continue the example $f = dS(\cdot)$ from above. Let $X \subset \mathbb{R}^n$ such that $TX = \mathbb{R}^n \times X$. Sections of TX are simply n-tuples of functions

$$s = \begin{bmatrix} s^1 \\ \vdots \\ s^n \end{bmatrix} \in M_{n\times 1}(C(X)) .$$

Then for $S \in C(X)$ any function we have globally

$$dS = [dS_1, \cdots , dS_n] \in M_{1\times n}(C(X)) .$$

The Tate construction: killing of cohomology groups

In practice it is often useful to resolve quotients $A/(E_1, \cdots, E_n)$ where the $(E_1, \cdots, E_n)$ do *not* form a regular sequence.
There is a canonical procedure, going back to John Tate (and used throughout rational homotopy theory, see the example below), to systematically "kill" all unwanted cohomology groups by introducing further generators.

Example

Let again

$$f : M \to A$$

be a morphism of A-modules with nontrivial kernel

$$\ker(f) \hookrightarrow M \ .$$

Then instead of using the 2-term complex

$$V := \ (0 \longrightarrow V^{-1} \xrightarrow{d_V^0} V^0 \longrightarrow 0)$$

$$(0 \longrightarrow M \xrightarrow{f} A \longrightarrow 0)$$

consider the 3-term complex

$$W := \left(0 \longrightarrow W^{-2} \xrightarrow{d_W^{-1}} W^{-1} \xrightarrow{d_W^{0}} W^{0} \longrightarrow 0\right) .$$

$$\left(0 \longrightarrow \ker(f) \hookrightarrow M \xrightarrow{f} A \longrightarrow 0\right)$$

anti ghosts	anti fields	fields

The introduction of the new term in degree -2 “kills” all unwanted cohomology in degree -1. Therefore, by construction, the cohomology of W is concentrated in degree 0

$$
\begin{array}{cccccccccc}
H(W) := & (0 & \longrightarrow & H(W)^{-2} & \xrightarrow{d^{-1}_{H(W)}} & H(W)^{-1} & \xrightarrow{d^{0}_{H(W)}} & H(W)^0 & \longrightarrow & 0) \;. \\
 & (0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A/\mathrm{im}(f) & \longrightarrow & 0)
\end{array}
$$

It might seem that forming now $\wedge^n W$ instead of $\wedge^n V$ produces a dg-algebra with no cohomology away from degree 0.

However, the Künneth formula tells us, for the cohomology of the tensor product of two complexes X and Y, that

$$0 \to \left(\bigoplus_{p+q=n} H(X)^p \otimes H^q(q)\right) \to H^n(X \otimes Y) \to \left(\bigoplus_{p+q=n+1} \mathrm{Tor}(H^p(X), H$$

is exact. This implies that even if the cohomologies of X and Y are both concentrated in degree 0, we get

$$H^{-1}(X \otimes Y) \simeq \mathrm{Tor}(H^0(X), H^0(Y)),$$

which may be nontrivial.

Example

Suppose that the non-positively graded complex V is a projective resolution of A/I in degree 0, $H^0(V) = A/I$, $H^{-n<0}(V) = 0$ for $I = (E_1, \cdots, E_n)$ some ideal generated by a regular sequence. For instance take $V = K[E_1, \cdots, E_n]$ to be the Koszul complex of that regular sequence.
Or, in our standard example, consider

$$V = \left(0 \longrightarrow V^{-2} \xrightarrow{d_{V_{-1}}^{-1}} V^{-1} \xrightarrow{d_V^0} V^0 \longrightarrow 0\right)$$

$$\left(0 \longrightarrow \ker(dS(\cdot)) \hookrightarrow \Gamma(TX) \xrightarrow{dS(\cdot)} C(X) \longrightarrow 0\right)$$

and assume that $\ker(dS(\cdot))$ is projective, i.e. sections of a vector bundle over X.

Then the following is true (see [Loday:Cyclic Homology, 3.4.7])

Fact

The first Tor-algebra is

$$\mathrm{Tor}^{-1}(A/I, A/I) \simeq I/I^2\,.$$

The higher Tor-algebras are the exterior powers of this:

$$\mathrm{Tor}^{-\bullet}(A/I, A/I) \simeq \wedge^{\bullet}_{A/I}(I/I/2)$$

Remark

If $A = C(X)$, then $\wedge^\bullet(I/I^2)$ is the algebra of differential forms on X restricted the vanishing set of $(E_1, \cdots, E_n)$.
So we find in this case that even though the cohomology of V is concentrated in degree 0, the cohomology of $V \otimes V$ can be nontrivial in degree -1.

The Tate construction

Let $(V, \cdot)$ be monoid in $\mathrm{Ch}^\bullet(A)$, hence a dg-algebra over A, concentrated in either non-positive or non-negative degree. Let us assume V is in non-positive degree for definiteness, as in our applications.

There is a systematic way to create from $(V, \cdot)$ a new dg-algebra $(V', \cdot)$ extending it

$$(V', \cdot) \longrightarrow\!\!\!\!\!\rightarrow (V, \cdot)$$

with the property that the cohomology of V' vanishes everywhere except in degree 0, where it coincides with the cohomology of V.

The procedure works by induction over the degree of the cohomology groups:

- Let $(V_{-k}, \cdot) \longrightarrow\!\!\!\!\!\rightarrow (V, \cdot)$ be a dg-algebra extending V such that
$$H^{-k<d<0}(V_{-k}) = 0\,.$$
- add an addition generator etc [need to rewrite this]

Using this procedure, one obtains the following

Fact (Tate)

For I any ideal in $\mathcal{A}$ there exists a free acyclic dg-algebra X such that $H^0(X) = \mathcal{A}/I$.

In other words: we can *always* find *some* resolution of a quotient A/I by a dg-algebra.

Remark.

In the context of the BV formalism, it is for this reason that one is actually not primarily interested in Koszul complexes themselves: even if they fail to provide a resolution, using the Tate construction ("incorporating (possibly higher order)antighosts"), one always forms a resolution of the "shell".

Example (rational homotopy groups of spheres).

Let $A = \mathbb{R}$ be the field of real numbers.
Suppose we want to build a graded-commutative dg algebra V with the only nontrivial cohomology group being $H^{2n+1}(V) = A$.
Clearly, this is simply achieved by letting V be generated from a single degree $2n+1$ generator ω

$$V = V^{2n+1} = \langle \omega \rangle$$

with $d\omega = 0$. Since,

$$\omega \wedge \omega = 0$$

due to the fact that $2n+1$ is odd, this choice is consistent and no further generators need to be introduced.

But now consider the same situation for even degree: suppose the graded-commutative dg-algebra V has a single non-exact degree $2n$-generator ω with $d\omega = 0$. Then the cohomology in degree $2n$ is again A. But now also all elements of the form $\omega \wedge \omega \wedge \cdots \omega$ are non-vanishing and closed. In order to remove the unwanted cohomology generated by these, we throw in another generator, λ, in degree $4n - 1$, and set

$$d\lambda = \omega \wedge \omega \,.$$

This removes all the superfluous cohomologies: now all troublesome elements are exact.

$$\omega \wedge \omega \wedge \underbrace{\omega \wedge \cdots \wedge \omega}_{k \in \mathbb{N}} = d(\lambda \wedge \underbrace{\omega \wedge \cdots \wedge \omega}_{k}) \,.$$

Notice that

- the nontrivial homology groups of the n-sphere are

$$H_k(S^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = n, 0 \\ 0 & else \end{cases}$$

- the *rational* homotopy groups of the n-sphere are

$$\pi_k(S^{2n+1}) = \begin{cases} \mathbb{Q} & k = 2n+1 \\ 0 & else \end{cases}$$

$$\pi_k(S^{2n}) = \begin{cases} \mathbb{Q} & k = 2n, k = 4n-1 \\ 0 & else \end{cases}$$

.

This matches the pattern which we found for complexes with cohomology in a single degree, under the identification

quasi-free dg algebra	**rational top. space**
degree k cohomology	degree k chomology
degree n generators	rational homotopy group in degree n

Table: **The relation between dg-algebra and topological spaces** in terms of rational cohomology and homotopy groups.

The full Batalin-Vilkovisky complex

Observation

The **Weil algebra** $\mathrm{W}(\mathfrak{g})$ of an L_∞-algebra $\mathfrak{g}$ is the same as what is addressed as the space of functions on the **shifted tangent bundle** of the supermanifold on which $\mathrm{CE}(\mathfrak{g})$ is the space of functions.

tangent category	inner automorphism $(n+1)$-group	inner derivation Lie $(n+1)$-algebra	Weil algebra	shifted tangent bundle

$$\mathrm{CE}(\mathrm{Lie}(T\mathbf{B}G)) = \mathrm{CE}(\mathrm{Lie}(\mathrm{INN}(G))) = \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) = \mathrm{W}(\mathfrak{g}) = C^\infty(T[1]X_{\mathfrak{g}})$$

Figure: **A remarkable coincidence of concepts** relates the notion of graded tangency to the notion of universal bundles. See [Roberts-S.] and [Sati-S.-Stasheff].

Recall from [Sati-S.-Stasheff] that the Weil algebra $\mathrm{W}(\mathfrak{g})$ of an L_∞-algebra $\mathfrak{g}$ plays the role of differential forms on the universal $\exp(\mathfrak{g})$-bundle:

	dg-algebras	(pointed) topological spaces	
Chevalley-Eilenberg algebra	$\mathrm{CE}(\mathfrak{g})$	G	structure group
	$\uparrow i^*$	$\downarrow i$	
Weil algebra	$\mathrm{W}(\mathfrak{g})$	EG	universal G-bundle
	$\uparrow p^*$	$\downarrow p$	
algebra of invariant polynomials	$\mathrm{inv}(\mathfrak{g})$	BG	classifying space for G

Definition: vertical and horizontal derivations

- **vertical derivations** on $\mathrm{W}(\mathfrak{g})$ are those vanishing on the shifted copy $\mathfrak{g}^*[1] \subset \mathrm{W}(\mathfrak{g})$
-
- **horizontal derivations** on $\mathrm{W}(\mathfrak{g})$ are those vanishing on the un-shifted copy $\mathfrak{g}^* \subset \mathrm{W}(\mathfrak{g})$

Observation

It follows that the BV-complex comes from the horizontal derivations on a configuration space object whose algebra of forms is a Weil algebra.

More details later.

previous: ∞-configuration spaces

Examples

- The archetypical example: the (-1)-brane.
- The crucial motivating example: ordinary gauge theory.
- The first example showing the full power: higher gauge theory.

next: annotated literature

Example: the (-1)-brane

The simple – but very instructive – case where configuration space is finite dimensional.

In practice the BV complex is applied in the context of local functions on some jet space.
But here we want to get rid of inessential technicalities as far as possible and try to extract the pure relevant structure. For that reason we study the BV complex in a setup where we are dealing with ordinary functions on some manifold.
Mathematically, this amounts to studying the critical points of an ordinary function by cohomological means. Physically it can fruitfully be thought of as the standard BV formalism applied to what is called a (-1)-brane: an object whose worldvolume is a single point.

Remark: the algebra of functions

Some of the crucial statements about the cohomologies of the complexes we are about to consider depend sensitively on the precise nature of the function algebras over which we work.
For X a (real) manifold, we shall write $C(X)$ for an unspeified class of functions on X as long as it does not matter. When it matters, we will choose from

$$C(X) \rightsquigarrow \left\{ \begin{array}{rl} C^\omega(X) & \text{real analytic functions} \\ C^\infty(X) & \text{smooth functions} \end{array} \right.$$

The ingredients

Definition

Let X be a smooth manifold and $S \in C^\infty(X)$ a smooth function on it. Denote by $\Gamma(TX)$ the space of smooth vector fields on X. Consider the 3-term chain complex

$$\begin{array}{ccccc} \ker(dS(\cdot)) & \hookrightarrow & \Gamma(TX) & \xrightarrow{dS(\cdot)} & C^\infty(X) \\ & & & & \\ -2 & & -1 & & 0 \end{array}$$

with the degrees as indicated. The corresponding 3-vector space we denote

$$WC(\Sigma) \in 3\mathrm{Vect}\,.$$

The physics terminology.

The entities in the above definition are known in the physics literature under the following names.

- The space X is the **configuration space** or the **space of histories** (the difference need not concern us here).
- An point $x \in X$ is a **field configuration**.
- The function S is the **action**.
- The condition $dS = 0$ is the **equations of motion**.
- The space of critical points $\Sigma := \{x \in X \,|\, (dS)_x = 0\}$ is the **shell**.
- The elements of the space $\ker(dS(\cdot))$ are the **Noether identities**.

The notation $WC(\Sigma)$ is for smooth functions on the "*w*eak *s*hell". By construction, the cohomology of $WC(\Sigma)$ is concentrated in degree 0, where it is $C(X)/\mathrm{im}(dS(\cdot))$. Following the situation in physical examples, we assume that $\mathrm{im}(dS(\cdot))$ is the space of all smooth functions on X that vanish on Σ. (** When is this assumption satisfied?? **)

Observation

Hence the above chain complex is a resolution for the space of on-shell functions, which is the way it is usually thought of. Passing from homotopical to n-categorical language this means: the 3-vector space $WC(\Sigma)$ is equivalent, as a 3-vector space, to the 1-vector space of on-shell functions.

Symmetries and Noether identities

We are now going to give what is supposed to be the standard definition of *symmetries* and *Noether identities* as they appear in physics, adapted to the toy example we are looking at, where X is a mere manifold.

Definition

Given $S \in C(X)$ as above, we say

- A (local) **symmetry** of S is a vector field $v \in \Gamma(TX)$ such that the Lie derivative

  $$\mathcal{L}_{\epsilon v} dS = 0$$

  for all $\epsilon \in C(X)$.
- A **Noether identity** of S is a vector field v such that

  $$\mathcal{L}_v S = 0 \,.$$

Remark.

To see the connection of this definition to the definitions one might find in most of the physics literature notice that

- The above says that a local symmetry preserves the equations of motion. This corresponds to the more common requirement that the local symmetry preserves the Lagrangian up to a divergence. Compare with p.7 of [KazinskiLyakhovichSharapov:1993].
- To see that a Noether identity can be regarded as a vector preserving the action in our context, take the usual defintion and truncate jet space at 0th order everywhere. Compare with [StasheffFulpLada:2002].

In the same vein, the following plays the role of **Noether's second theorem** in the context of our toy example.

Proposition: toy version of Noether's second theorem

The space of local symmetries is isomorphic to that of Noether identities.

Proof. Using Cartan's "magic formula" we have

$$\begin{aligned} \mathcal{L}_{\epsilon v} dS &= \epsilon \mathcal{L}_v dS + d\epsilon \wedge v(S) \\ &= \epsilon d(v(S)) + v(S) d\epsilon \,. \end{aligned}$$

Clearly, every Noether identity v is hence also a local symmetry. Conversely, if v is a local symmetry then from $\mathcal{L}_v dS = 0$ and using $\mathcal{L}_{\epsilon v} dS = 0$ in the above formula for all $\epsilon \in C(X)$ it follows that v is a Noether identity. $\square$

chain complex $V =$

$$\begin{array}{ccccccc} \ker(dS(\cdot)) & \hookrightarrow & \Gamma(TX) & \xrightarrow{dS(\cdot)} & C(X) & \xrightarrow{d_\rho} & C(X)\otimes\mathfrak{g}^* \\ \| & & & & & & \\ C(X)\otimes\mathfrak{g} & & & & & & \end{array}$$

degree		-2	-1	0	1
		⇐ass. 3-algebra⇒			⇐Lie 1-algebra⇒
Noether's second thm.		Noether identities			local symmetries
math guys		Tate	Koszul		Chevalley-Eilenberg
physics names		*antighosts*	*antifields*	*fields*	*ghosts*
antifield number	= **ω-vector space dimension**	2	1	0	
ghost number	= **ω-covector space dimension**				1

Local symmetry Lie algebras

We now make the assumption that we have a "Lie algebra of local gauge symmetries".

Definition

In the case that the space of local symmetries of S is of the form $C(X) \otimes \mathfrak{g}$ with $\mathfrak{g}$ a finite-dimensional Lie algebra equipped with an action

$$\rho : \mathfrak{g} \to \Gamma(TX)$$

we say that $\mathfrak{g}$ is the **gauge Lie algebra** of S and that ρ are the **gauge transformations of the fields**.

In that case we can extend our 3-term complex $WC(\Sigma)$ by the map

$$C(X) \xrightarrow{d_\rho} C(X) \otimes \mathfrak{g}^*$$

by setting

$$d_\rho : f \mapsto \rho(\cdot)(f) .$$

Remark

Notice that d_ρ is the differential of the Chevalley-Eilenberg complex that computes the Lie algebra cohomology of $\mathfrak{g}$ with values in the Lie module $C(X)$ restricted to degree 0.

Example: the case where X is a principal bundle

In the more well-behaved situations the local symmetries will act freely on our space X, and X will be a principal G-bundle.
So assume that the Lie group G acts on X such that $p : X \to X/G$ is a principal G-bundle. Let

$$S \in C(X)$$

be the pullback of a smooth function $S_G \in C(X/G)$ downstairs

$$S := p^* S_G$$

which has the property that it is not annihilated by any nontrivial vector field on X/G.

Then the local symmetries of S are precisely the *vertical* vector fields on the G-bundle X, namely sections

$$\Gamma_{\mathrm{vert}}(TX) := \Gamma(\mathrm{Vert}(X)) \simeq C(X) \otimes_{\mathbb{R}} \mathfrak{g}$$

of the vector bundle of vertical vector fields

$$\mathrm{Vert}(X) := \ker(dp) \simeq P \times \mathfrak{g}\,.$$

So in this case our complex is

$$(0 \longrightarrow B^{-2} \longrightarrow B^{-1} \longrightarrow C(X) \longrightarrow A \otimes \mathfrak{g} \longrightarrow 0)\ .$$

$$(0 \longrightarrow \Gamma_{\mathrm{vert}}(TX) \hookrightarrow \Gamma(TX) \xrightarrow{dS(\cdot)} C(X) \xrightarrow{d_\rho} C(X) \otimes_{\mathbb{R}} \mathfrak{g} \longrightarrow 0)$$

invariant functions on the trivial circle bundle

As a simple special case, consider the following example, which models essentially the harmonic oscillator with a circle worth of gauge degeneracies thrown in:
Let

$$X = \mathbb{R} \times S^1$$

the cylinder, thought of as the trivial circle bundle

$$p : X \to \mathbb{R}$$

and let the action $S \in C^\omega(X)$ be

$$S = p^*(x \mapsto x^2)\,.$$

Then

$$\ker(dS) = \ker(dp)$$

are the analytic vertical vector fields on S^1.

Since dS is just multiplication of (component) functions by x, we find that the on-shell functions are indeed precisely the quotient

$$C^\omega(\Sigma) \simeq \mathrm{coker}(dS(\cdot)) = C^\omega(X)/\mathrm{im}(dS(\cdot))\,.$$

Since furthermore

$$\ker(d_\rho) \simeq C^\omega(X/G)$$

we find that the cohomology of the above complex in degree 0 is precisely that of gauge-invariant on-shell functions

$$C^\omega(\Sigma/G) = C^\omega(\{0\}) = \mathbb{R}\,.$$

Example: ordinary gauge theory

The case where configuration space is the space of Lie-algebra valued forms on some manifold.

The setup

For us here, a gauge theory is a field theory whose **configurations are connections** ∇ on principal G-bundles $P \to Y$ over some space Y, for G some Lie group.
To start with, we restrict attention to *trivial* principal bundles over X. In that case, a connection is precisely a Lie-algebra valued 1-form

$$A \in \Omega^\bullet(Y, \mathfrak{g}),$$

with $\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra corresponding to G.
We will now analyze the space of all such Lie algebra valued forms *internally* and show that we thereby automatically obtain the structure of the corresponding BV-complex, complete with fields, ghosts, antifields and antighosts.

Let $\mathfrak{g}$ be any (finite dimensional) Lie ∞-algebra, $\mathrm{CE}(\mathfrak{g})$ its Chevalley-Eilenberg DGC algebra and $\mathrm{W}(\mathfrak{g})$ its Weil DGCA. For Y any smooth space, $\mathfrak{g}$-valued differential forms on Y are DGCA morphisms

$$\Omega^\bullet(Y) \xleftarrow{(A,F_A)} \mathrm{W}(\mathfrak{g}) \; .$$

So the *set* of $\mathfrak{g}$-valued differential forms is

$$\mathrm{Hom}_{\mathrm{DGCA}s}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y)) \, .$$

We want to consider the algebra of differential forms on the *smooth space* of $\mathfrak{g}$-valued forms on Y:

$$\mathrm{maps}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y))$$

according to Differential forms on spaces of maps.

the configuration space of ordinary gauge theory

Let $\mathfrak{g}$ be an ordinary Lie algebra and Y be some manifold. The configuration space of ordinary $\mathfrak{g}$-gauge theory (assuming trivial bundles for the moment) is

$$\Omega^\bullet(Y, \mathfrak{g}) := \mathrm{Hom}_{\mathrm{DGCA}s}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y)) .$$

We now analyze the algebra

$$\mathrm{maps}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(\mathfrak{g}))$$

and demonstrate that it is itself the Weil algebra of some Lie 2-algebroid.

To make contact with the physics literature and most of the BV-literature, we describe everything in components.
So let $Y = \mathbb{R}^n$ and let $\{x^\mu\}$ be the canonical set of coordinate functions on Y. Choose a basis $\{t_a\}$ of $\mathfrak{g}$ and let $\{t^a\}$ be the corresponding dual basis of $\mathfrak{g}^*$. Denote by

$$\delta_y \iota_{\frac{\partial}{\partial x^\mu}}$$

the delta-current on $\Omega^\bullet(Y)$ which sends a 1-form ω to

$$\omega_\mu(y) := \omega(\frac{\partial}{\partial x^\mu})(y)\,.$$

Summary of the main result.

Recall that the Weil algebra $\mathrm{W}(\mathfrak{g})$ is generated from the $\{t^a\}$ in degree 1 and the σt^a in degree 2, with the differential defined by

$$dt^a = -\frac{1}{2}C^a{}_{bc}t^b \wedge t^c + \sigma t^a$$

$$d(\sigma t^a) = -C^a{}_{bc}t^b \wedge (\sigma t^c)\,.$$

We will find that $\mathrm{maps}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y))$ does look pretty much entirely like this, only that all generators are now forms on Y.

fields	$\left\{A^a_\mu(y), (F_A)_{\mu\nu}(y) \in \Omega^0(\Omega(Y,\mathfrak{g})) \mid y \in Y, \mu,\nu \in \{1,\cdots,\dim(Y), a \in \{1,\ldots,\dim(\mathfrak{g})\}\}\right\}$
ghosts	$\left\{c^a(y)\Omega^1(\Omega(Y,\mathfrak{g})) \mid y \in Y, a \in \{1,\ldots,\dim(\mathfrak{g})\}\}\right\}$
antifields	$\left\{\frac{\partial}{\partial \delta A^a_\mu(y)} \in \mathrm{Hom}(\Omega^1(\Omega(Y,\mathfrak{g})),\mathbb{R}) \mid y \in Y, \mu \in \{1,\cdots,\dim(Y), a \in \{1,\ldots,\dim(\mathfrak{g})\}\}\right\}$
anti-ghosts	$\left\{\frac{\partial}{\partial \beta^a(y)} \in \mathrm{Hom}(\Omega^2(\Omega(Y,\mathfrak{g})),\mathbb{R}) \mid y \in Y, \dim(Y), a \in \{1,\ldots,\dim(\mathfrak{g})\}\}\right\}$

Table: **The BV field content of gauge theory** obtained from our almost internal hom of dg-algebras, definition **??**. The dgc-algebra $\mathrm{maps}(\mathrm{W}(\mathfrak{g}),\Omega^\bullet(Y))$ is the algebra of differential forms on a smooth space of maps from Y to the smooth space underlying $\mathrm{W}(\mathfrak{g})$. In the above table β is a certain 2-form that one finds in this algebra of forms on the space of $\mathfrak{g}$-valued forms.

Remark.

Before looking at the details of the computation, recall that an n-form ω in $\mathrm{maps}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y))$ is an assignment

$$\begin{tikzcd} U \arrow[d, "\phi"'] & \mathrm{Hom}_{\mathrm{DGCAs}}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y \times U)) \arrow[r, "\omega_U"] & \Omega^\bullet(U) \\ V & \mathrm{Hom}_{\mathrm{DGCAs}}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y \times V)) \arrow[u, "\phi^*"] \arrow[r, "\omega_V"] & \Omega^\bullet(V) \arrow[u, "\phi^*"'] \end{tikzcd}$$

of forms on U to $\mathfrak{g}$-valued forms on $Y \times U$ for all manifolds U, conatural in U.

We concentrate on those n-forms ω which arise in the way of proposition **??**.

0-Forms

The 0-forms on the space of $\mathfrak{g}$-value forms are constructed as in proposition **??** from an element $t^a \in \mathfrak{g}^*$ and a current $\delta_y \iota_{\frac{\partial}{\partial x^\mu}}$ using

$$t^a \delta_y \iota_{\frac{\partial}{\partial x^\mu}}$$

and from an element $\sigma t^a \in \mathfrak{g}^*[1]$ and a current

$$\delta_y \iota_{\frac{\partial}{\partial x^\mu}} \iota_{\frac{\partial}{\partial x^\nu}} \,.$$

0-Forms

This way we obtain the families of functions (0-forms) on the space of $\mathfrak{g}$-valued forms:

$$A^a_\mu(y) : (\Omega^\bullet(Y \times U) \leftarrow \mathrm{W}(\mathfrak{g}) : A) \mapsto (u \mapsto \iota_{\frac{\partial}{\partial x^\mu}} A(t^a)(y,u))$$

and

$$F^a_{\mu\nu}(y) : (\Omega^\bullet(Y \times U) \leftarrow \mathrm{W}(\mathfrak{g}) : F_A) \mapsto (u \mapsto \iota_{\frac{\partial}{\partial x^\mu}} \iota_{\frac{\partial}{\partial x^\nu}} F_A(\sigma t^a)(y,u))$$

which pick out the corresponding components of the $\mathfrak{g}$-valued 1-form and of its curvature 2-form, respectively.
This are the *fields*.

1-Forms

A 1-form on the space of $\mathfrak{g}$-valued forms is obtained from either starting with a degree 1 element and contracting with a degree 0 delta-current

$$t^a \delta_y$$

or starting with a degree 2 element and contracting with a degree 1 delta current:

$$(\sigma t^a)\delta_y \frac{\partial}{\partial x^\mu} .$$

1-Forms

To get started, consider firstt the case where $U = I$ is the interval. Then a DGCA morphism

$$(A, F_A) : \mathrm{W}(\mathfrak{g}) \to \Omega^\bullet(Y) \otimes \Omega^\bullet(I)$$

can be split into its components proportional to $dt \in \Omega^\bullet(I)$ and those not containing dt.
We hence can write the general $\mathfrak{g}$-valued 1-form on $Y \times I$ as

$$(A, F_A) : t^a \mapsto A^a(y,t) + g^a(y,t) \wedge dt$$

and the corresponding curvature 2-form as

$$(A, F_A) : \sigma t^a \mapsto (d_Y + d_t)(A^a(y,t) + g^a(y,t) \wedge dt) + \frac{1}{2} C^a{}_{bc}(A^a(y,t) + g^a(y,$$

$$= F_A^a(y,t) + (\partial_t A^a(y,t) + d_Y g^a(y,t) + [g, A]^a) \wedge dt\,.$$

By contracting this again with the current $\delta_y \frac{\partial}{\partial x^\mu}$ we obtain the 1-forms

$$t \mapsto g^a(y,t)dt$$

and

$$t \mapsto (\partial_t A^a_\mu(y,t) + \partial_\mu g^a(y,t) + [g, A_\mu]^a)dt$$

on the interval.

1-Forms

We will identify the first one with the component of the 1-forms on the space of $\mathfrak{g}$-valued forms on Y called the *ghosts* and the second one with the 1-forms which are killed by the derivations called the *anti-fields*.

To see more of this structure, consider now $U = I^2$, the unit square.

Then a DGCA morphism

$$(A, F_A) : \mathrm{W}(\mathfrak{g}) \to \Omega^\bullet(Y) \otimes \Omega^\bullet(I^2)$$

can be split into its components proportional to $dt^1, dt^2 \in \Omega^\bullet(I^2)$.

1-Forms

We hence can write the general $\mathfrak{g}$-valued 1-form on $Y \times I$ as

$$(A, F_A) : t^a \mapsto A^a(y,t) + g_i^a(y,t) \wedge dt^i \,,$$

and the corresponding curvature 2-form as

$$(A, F_A) : \sigma t^a \mapsto (d_Y + d_{I^2})(A^a(y,t) + g_i^a(y,t) \wedge dt^i + h^a(y,t) dt^1 \wedge dt^2)$$

$$+\frac{1}{2} C^a{}_{bc}(A^a(y,t) + g_i^a(y,t) \wedge dt^i + h^a(y,t) dt^1 \wedge dt^2) \wedge (A^b(y,t) + g_i^b(y,t) \wedge d$$

$$= F_A^a(y,t) + (\partial_{t^i} A^a(y,t) + d_Y g_i^a(y,t) + [g_i, A]^a) \wedge dt^i$$

$$+ (\partial_i g_j^a + [g_i, g_j]^a) dt^i \wedge dt^j \,.$$

1-Forms

By contracting this again with the current $\delta_y \frac{\partial}{\partial x^\mu}$ we obtain the 1-forms

$$t \mapsto g_i^a(y,t)dt^i$$

and

$$t \mapsto (\partial_t A_\mu^a(y,t) + \partial_\mu g_i^a(y,t) + [g_i, A_\mu]^a)dt^i$$

on the unit square.
This are again the local values of our

$$c^a(y) \in \Omega^1(\Omega^\bullet(Y,\mathfrak{g}))$$

and

$$\delta A_\mu^a(Y) \in \Omega^1(\Omega^\bullet(Y,\mathfrak{g}))\,.$$

1-Forms

The second 1-form vanishes in directions in which the variation of the $\mathfrak{g}$-valued 1-form A is a pure gauge transformation induced by the function g^a which is measured by the first 1-form.
Notice that it is the sum of the exterior derivative of the 0-form $A^a_\mu(y)$ with another term.

$$\delta A^a_\mu(y) = d(A^a_\mu(y)) + \delta_g A^a_\mu(y)\,.$$

The first term on the right measure the change of the connection, the second subtracts the contribution to this change due to gauge transformations. So the 1-form $\delta A^a_\mu(y)$ on the space of $\mathfrak{g}$-valued forms vanishes along all directions along which the form A is modfied purely by a gauge transformation.
The $\delta A^a_\mu(y)$ are the 1-forms the derivations dual to which will be the *antifields*.

2-Forms

We have already seen the 2-form appear on the standard square. We call this 2-form

$$\beta^a \in \Omega^2(\Omega^\bullet(Y, \mathfrak{g})),$$

corresponding on the unit square to the assignment

$$\beta^a : (\Omega^\bullet(Y \times I^2) \leftarrow \mathrm{W}(\mathfrak{g}) : A) \mapsto (\partial_i g_j^a + [g_i, g_j]^a) dt^i \wedge dt^j .$$

There is also a 2-form in the game, coming from $(\sigma t^a)\delta_y$.
Then one immediately sees that our forms on the space of $\mathfrak{g}$-valued forms satisfy the relations

$$dc^a(y) = -\frac{1}{2} C^a{}_{bc} c^b(y) \wedge c^c(y) + \beta^a(y)$$

and

$$d\beta^a(y) = -C^a{}_{bc} c^a(y) \wedge c^b(y) .$$

2-Forms

The 2-form β on the space of $\mathfrak{g}$-valued forms is what is being contracted by the horizontal derivations called the *antighosts*.
We see, in total, that $\Omega^\bullet(\Omega^\bullet(Y, \mathfrak{g}))$ is the Weil algebra of a DGCA, which is obtained from the above formulas by setting $\beta = 0$ and $\delta A = 0$. This DGCA is the algebra of the gauge groupoid, that where the only morphisms present are gauge transformations.
I just did this computation here over $U = I^2$. But I think it is clear how the computation generalizes and that this result is indeed true.

Example: higher gauge theory

The case where configuration space is the space of Lie ∞-algebra valued forms on some manifold.

A field configuration of a higher gauge theory is a connection of a principal $G_{(n)}$ n-bundle, for $G_{(n)}$ some Lie n-group. The most familiar example is the structure 2-group $G_{(n)} = G_{(2)} = \mathbf{B}U(1)$ which yields abelian gerbes.

As before, we start by concentrating on the case that the n-bundle in question is actually trivial. In that case a connection is precisely a differential form datum with values in the Lie n-algebra $\mathfrak{g} := \mathrm{Lie}(G_{(n)})$, as described in [Sati-S.-Stasheff].

So the discussion is completely analogously to the example Ordinary gauge theory, only that we use the Weil algebra $\mathrm{W}(\mathfrak{g})$ of the Lie ∞-algebra $\mathfrak{g}$ wherever we had used before the Weil algebra of an ordinary Lie algebra.

For a Lie n-algebra we find k-fold ghosts-of-ghosts and their corresponding antifields for $1 \leq k \leq n$.
The configuration space of a higher gauge theory with structure Lie n-algebra $\mathfrak{g}$ is a Lie n-groupoid. See also [Baez-S., S.-Waldorf].

previous: Examples

Annotated literature

A completely incomplete and highly subjective selection of some background literature, together with some remarks.

- Glenn Barnich
  Algebraic structure of gauge systems: Theory and Applications
  pdf slides for a talk at Bedlewo, October 19, 2007 available at
  `http://homepages.ulb.ac.be/ gbarnich/bedlewo.pdf`

  This is one of the better physics-style overviews of BV-formalism in quantum field theory. The items on slide number 3 correspond to our example The (-1)-brane.

Mathematical insight into BV-formalism

- Jim Stasheff
  The (secret?) homological algebra of the Balalin-Vilkovisky approach
  [http://arxiv.org/abs/hep-th/9712157]

  In this article the well-known homological algebraic structures underlying the BV-formalism are explicitly pointed out, more or less for the first time apparently. The Chevalley-Eilenberg complex and the Koszul-Tate complex sitting inside the BV-complex are pointed out, and the relation to L_∞-algebra, which I mentioned in Lie ∞-algebras, is mentioned. Follow the references given there.

Conceptual insights on the BV-formalism

The following three articles on BV-formalism stand out in that they indicate a deeper meaning underlying the formalism, which we discuss in Differential forms on spaces of maps:

Conceptual insights into the BV-formalism

- Alexandrov, Kontsevich, Schwarz and Zaboronsky
  The Geometry of the Master Equation and Topological Quantum Field Theory
  [http://arxiv.org/refs/hep-th/0608150]

  Reviewed in

  D. Roytenberg
  AKSZ-BV Formalism and Courant Algebroid-induced Topological Field Theories
  [http://arxiv.org/abs/hep-th/0608150]

  This develops the idea that configuration space conf has to be formed as something close to the *internal hom* in dg-algebras. See Differential forms on spaces of maps.

Conceptual insights into the BV-formalism

- E. Witten
  A note on the antibracket formalism
  Modern Physics Letters A, 5 7, 487-494

  This shows that the BV-Laplacian is nothing but the exterior differential acting on a non-standard Clifford module.

The following references are to the articles which the discussion of gauge theory makes use of and the discussion of higher gauge theory relies on.

Aspects of higher gauge theory

- J. Baez and U. S.
  Higher Gauge theory
  in Contemporary Mathematics, 431,
  Categories in Algebra, Geometry and Mathematical Physics
  available as [arXiv:math/0511710]

  This article discusses the parallel transport of Lie 2-algebra valued connections for possibly nontrivial 2-bundles. In the special case the the Lie 2-group is an automorphism 2-group, the differential 2-cocycles thus obtained reproduce those for nonabeliam gerbes with connection.

Aspects of higher gauge theory

- D. Roberts and U. S.
  The inner automorphism 3-group of a strict 2-group
  [*arXiv:0708.1741*]

  This article describes the Lie 3-group obtained from integrating the Lie 3-algebra $\mathrm{inn}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ for $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ a strict Lie 2-algebra. Notice that

  $$\mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) = \mathrm{CE}(\mathrm{inn}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}))\,.$$

  The article closes by indicating in which sense the inner automorphism 3-group is the univseral 2-bundle of the corresponding strict 2-group. This is the integrated analog of the fact we use here: that $\mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ plays the role of forms on the universal $\exp(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$2-bundle.

Aspects of higher gauge theory

- H. Sati, U. S. and J. Stasheff
  Lie ∞-algebra connections and application to String- and Chern-Simons n-transport
  [http://www.math.uni-hamburg.de/home/schreiber/LCon.pdf

  This article describes a way to handle Lie ∞-algebra $\mathfrak{g}$-connections entirely at the level of Lie ∞-algebras. The notion of Lie ∞-algebra valued forms used here, as well as the crucial interpretation of $\mathrm{W}(\mathfrak{g})$ as forms on the universal $\exp(\mathfrak{g})$-bundle is discussed there.

Aspects of higher gauge theory

- U. S. and K. Waldorf
  The geometry of smooth 2-functors
- U.S. and K. Waldorf
  Parallel transport and 2-functors