2D TFT from 2-Transport

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Abstract

It is shown that the parallel surface transport given by certain locally trivialized 2-functors from 2-paths to **Vect** reproduces the class of 2-dimensional topological field theories introduced by Fukuma, Hosono and Kawai. In general, every full 2-trivialization of a transport 2-functor gives rise to a Frobenius algebra bundle.

(Unfinished and unscrutinized private draft.)

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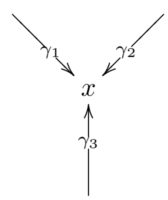
1 Introduction

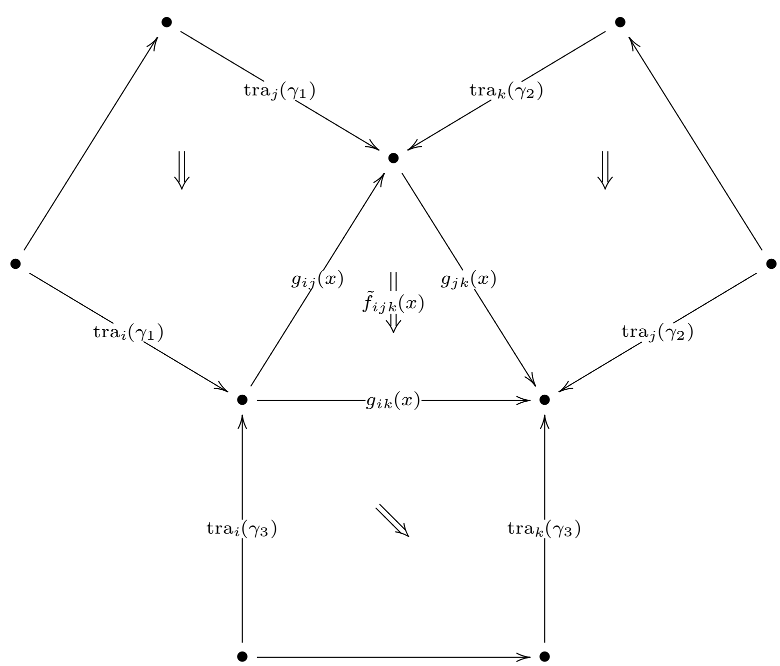
Parallel transport in a bundle with connection is a functor from paths to torsor morphisms. As a categorification of this fact, 2-functors from geometric 2-categories whose 2-morphisms are surface elements can be addressed as **parallel surface transport** or **2-transport**.

1.1 Motivation

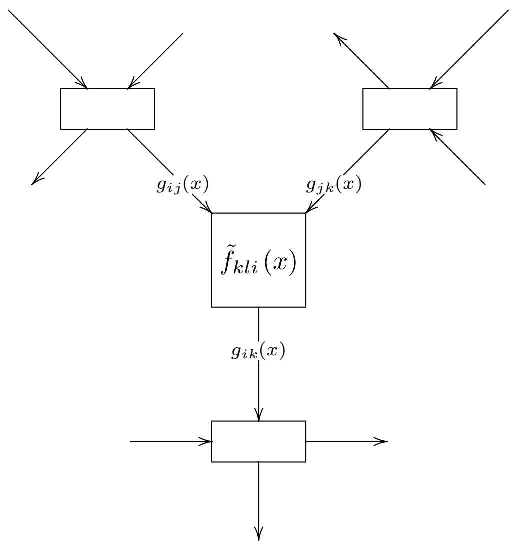
Like in ordinary parallel transport, one may consider **locally trivializing** a 2-transport functor. In [11] it was shown (and is rederived in the following) that locally trivialized surface transport acts on triangulations of a surface such that trivalent vertices



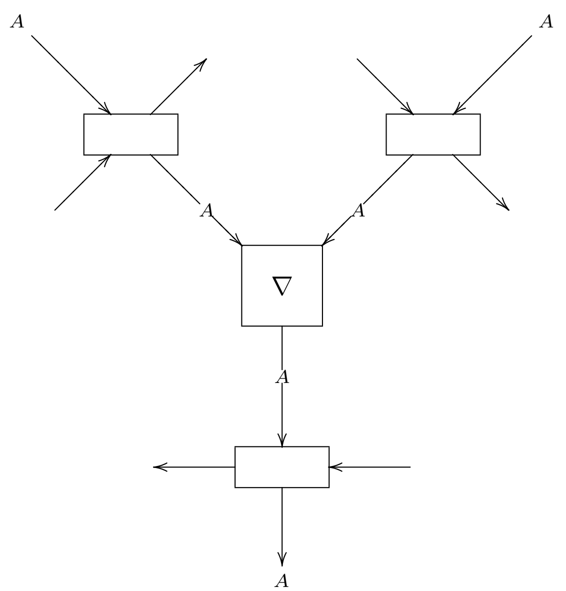
are decorated by 2-morphisms roughly like this:



Passing to the *dual* of this graph yields a graph of the form



In the case where all labels are independent of position x and furthermore all the labels on dual edges coincide, this has the form



Such decorations of triangulations by objects and morphisms in certain categories appear in the algebraic description of 2-dimensional (conformal/topological) field theory [6, 7]. In particular, if A is a vector space and ∇ an algebra product on this vector space, the above decoration of triangulations is associated to a 2-dimensional topological field theory of the kind introduced in [1].

The goal of the following discussion is to make these observations precise.

1.2 Outline

We consider transport 2-functors

$$\mathrm{tra} : \mathcal{P}_2(M) \to \mathbf{Vect}$$

from a geometric 2-category $\mathcal{P}_2(M)$ of surface elements ("2-paths") to the category of vector spaces **Vect**. (Like any monoidal category, **Vect** may be regarded as a 2-category with a single object.) For the special case where only 1-dimensional vector spaces are involved such 2-functors have been studied in [9], where it was shown that they are related to abelian bundle gerbes with connective structure.

Local trivializations of transport 2-functors with values in the 2-category of 2-torsors over some (strict) 2-group have been studied in section 12.4 of [11]. The constructions given there directly generalize to 2-functors with target **Vect**, with only slight modifications due to the fact that **Vect** is not a (2-)groupoid, but a monoidal category with duals.

One important difference introduced by this slight generalization, however, is that a locally trivialized 2-functor to **Vect** gives not just rise to a product, but also to a nontrivial coproduct structure, such that a Frobenius property is satisfied. This is the content of §2.

Given a local trivialization of a transport 2-functor, its total surface transport can be re-expressed in terms of a composition of local transports over faces of a chosen triangulation of the total surface. Doing so introduces certain transition data on the vertices and edges of the triangulation. This are the triangulation decorations mentioned above. These transitions are worked out in §3. This is done efficiently by passing back and forth between 2-morphism diagrams in **Vect** regarded as a 2-category and their dual (tangle-) diagrams in **Vect** regarded as a monoidal 1-category.

From the nature of these transitions one can easily read off the special case which reduces them to the triangulation decorations used in [1] for the description of 2-dimensional TFT. This is discussed in §4.

We could replace **Vect** by any other monoidal category without affecting the main points of our discussion. In the context of bundle gerbes one interesting choice is to replace **Vect** with $\mathbf{BiTor}(H)$, the monoidal category of bitorsors over some group H. Transport 2-functors with target $\mathbf{BiTor}(H)$ are related to *non*abelian bundle gerbes with connective structure [10]. However, in order to make contact with the algebraic description of CFT [6, 7] one would probably want to replace **Vect** with a category of representations of some chiral algebra.

Our discussion makes crucial use of the "tin can equations" that come with

1- and 2-morphisms of 2-functors. The reader can find the relevant definitions summarized in the appendix to [9].

1.3 Frobenius Algebras and Adjunctions

The relation between 2-trivializations and Frobenius algebras to be discussed in the following is actually a special realization of a general relation between Frobenius algebras and *adjunctions* [3]. Essentially, a 2-trivialization of a 2-functor realizes an adjunction on the trivialization 1-morphisms.

2 2-Trivializations

Definition 1 *Let*

$$\mathrm{tra} : \mathcal{P}_2(M) \to \mathbf{Vect}$$

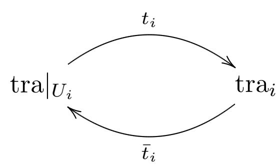
be a transport 2-functor on M. A (full) **local 2-trivialization** *of* tra *is*

1. *a choice of* **good covering** $\left(\bigsqcup_{i \in I} U_i = \mathcal{U}\right) \longrightarrow M$

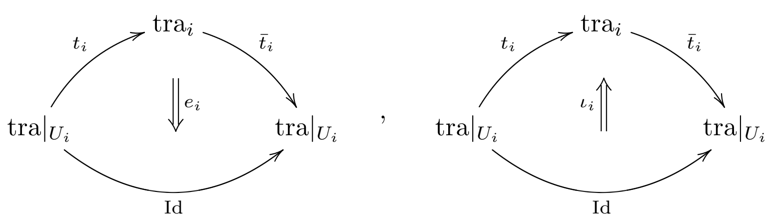
2. *for each $i \in I$ a choice of* **local transport 2-functor**

$$\mathrm{tra}_i : \mathcal{P}_2(U_i) \to \mathbf{Vect}$$

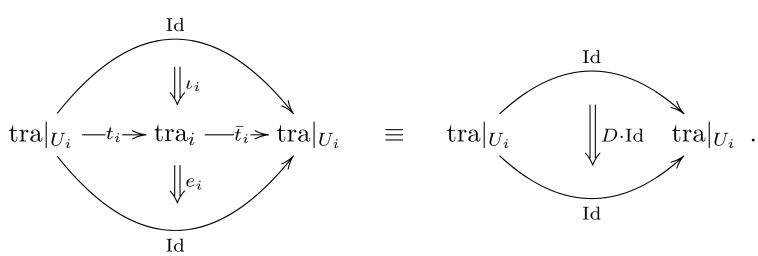
together with a choice of **local trivialization 1-morphisms**



and choices of 2-morphisms



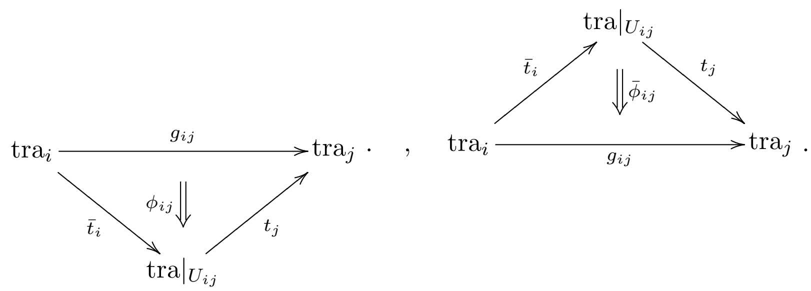
such that



3. *for all $i, j \in I$ a choice of* **trivialization transition 1-morphism**

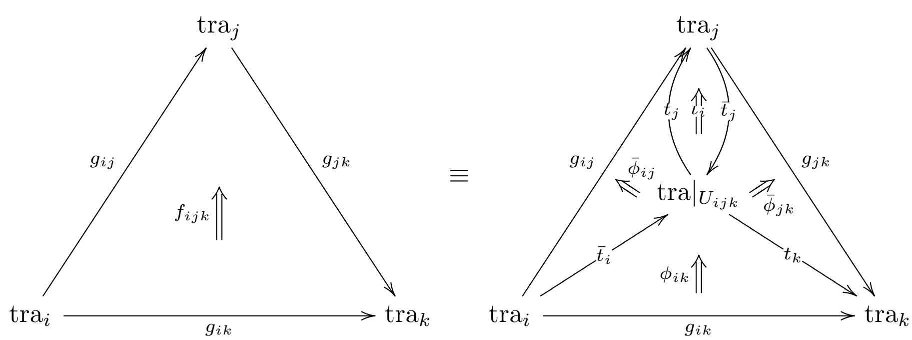
$$\mathrm{tra}_i \xrightarrow{\ g_{ij}\ } \mathrm{tra}_j$$

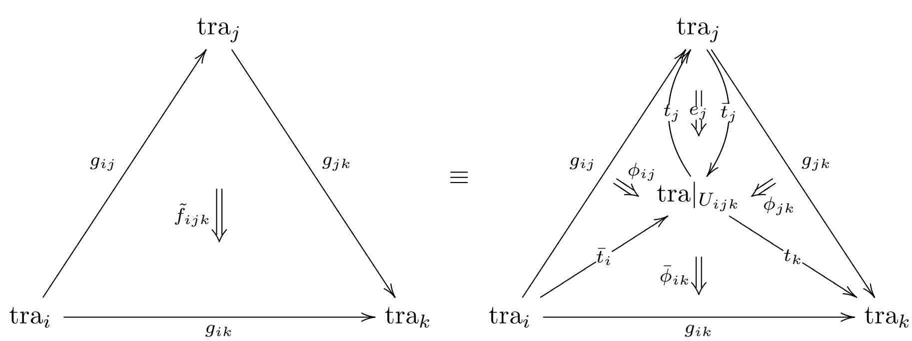
together with a choice of **local 2-trivialization** *2-morphisms*



For all $i \neq j$ *the 2-morphisms* ϕ_{ij} *and* ϕ_{ij} *are required to be 2-isomorphisms and to be their mutual inverses.*

Definition 2 *Associated to any local 2-trivialization of a transport 2-functor* tra *as above are, for all* $i, j, k \in I$, **trivialization 2-transition** *2-morphisms:*

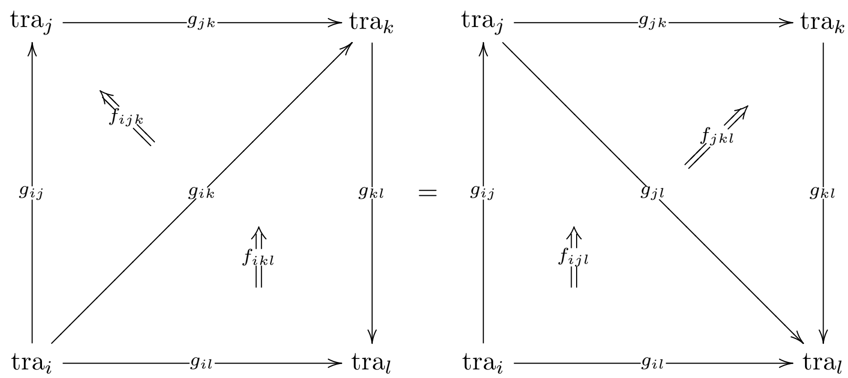




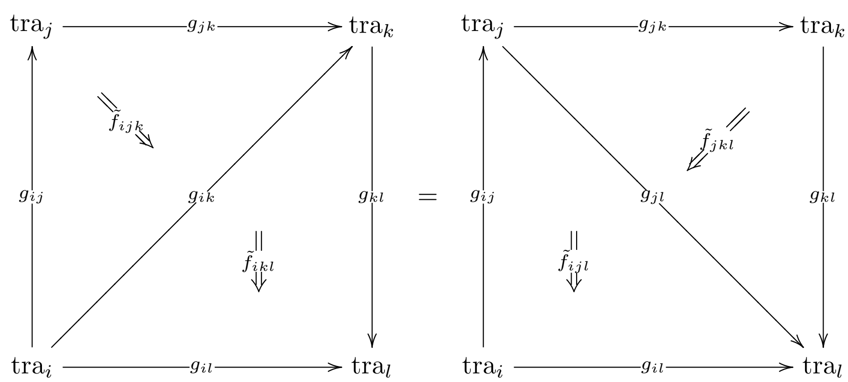
Remark. f_{ijk} and $\tilde{f}_{ijk}$ are not necessarily mutually inverse because e_i and ι_i need not be mutually inverse in general and ϕ_{ij}, $\bar{\phi}_{ij}$ are mutually inverse only for $i \neq j$.

Proposition 1 *For* $i, j, k \in I$ *pairwise distinct,* f_{ijk} *and* $\tilde{f}_{ijk}$ *behave like associative product and coproduct with Frobenius property. More precisely, the following equations hold for all pairwise distinct* $i, j, k, l \in I$*:*

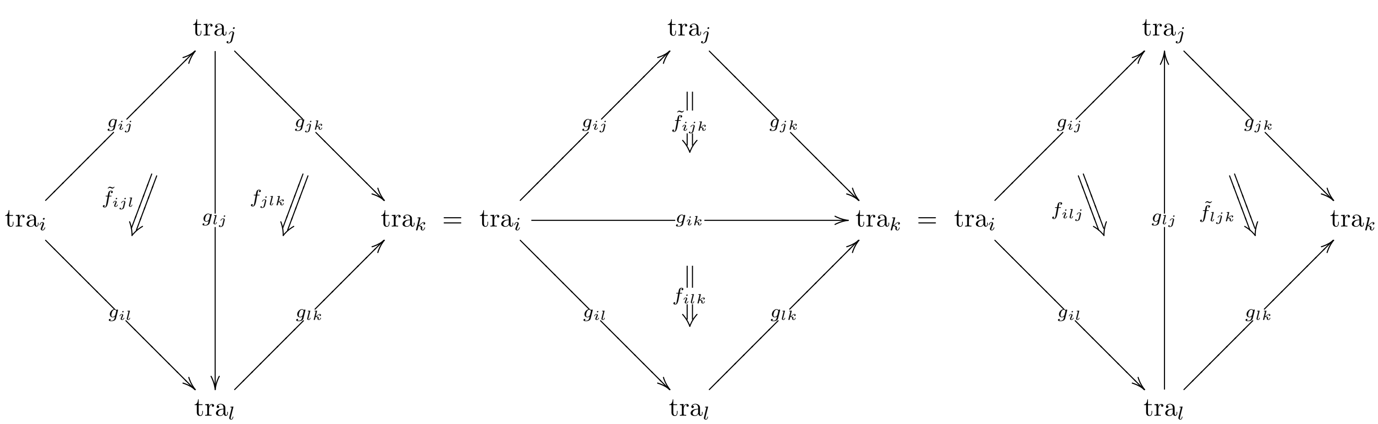
1. associativity of the coproduct



2. associativity of the product

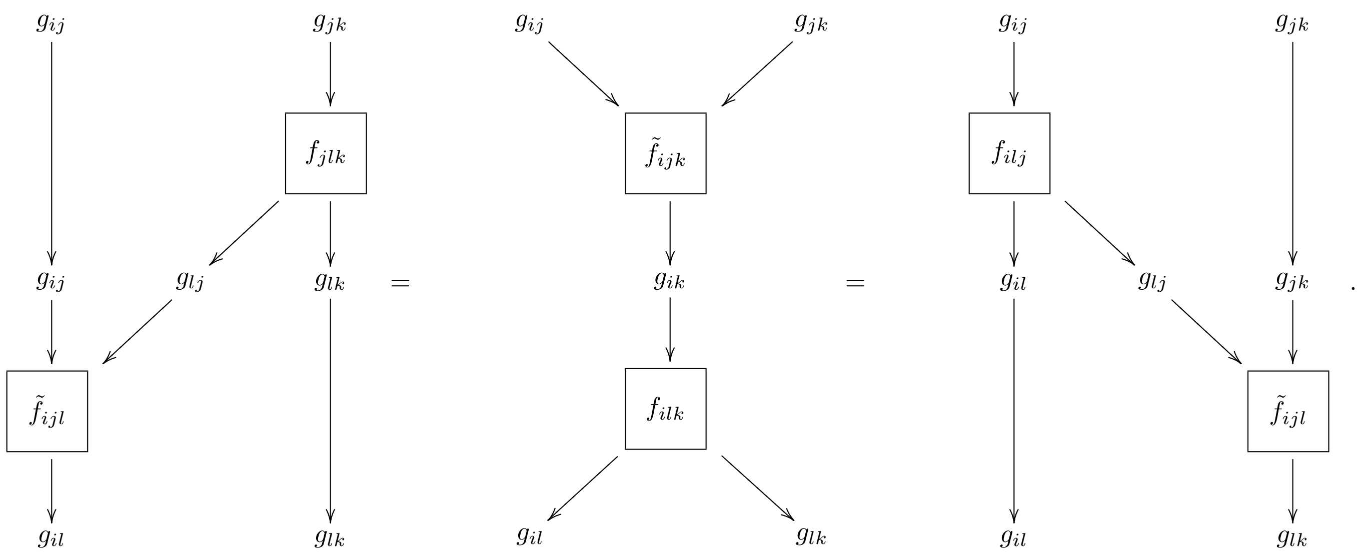


3. Frobenius property



Proof. The proofs of these three items are simple variations of the proof of the tetrahedron law given in [9]. □

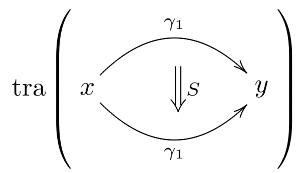
The reason for calling these equations "associativity" and "Frobenius property" becomes apparent when the above diagrams are expressed in terms of their *dual* graphs. For instance the third item then looks like



Switching to dual graphs will be shown below to manifestly relate structures appearing in 2-transport with well known structures in algebraic field theory.

3 Computing Locally Trivialized Surface Transport

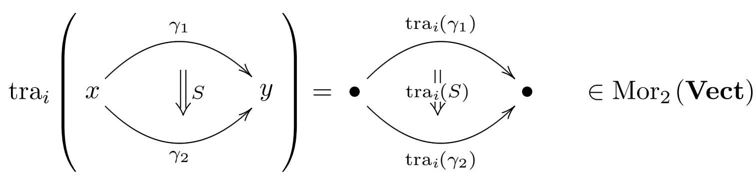
The existence of a local trivialization allows to express the transport



along a surface S which sits entirely inside U_i

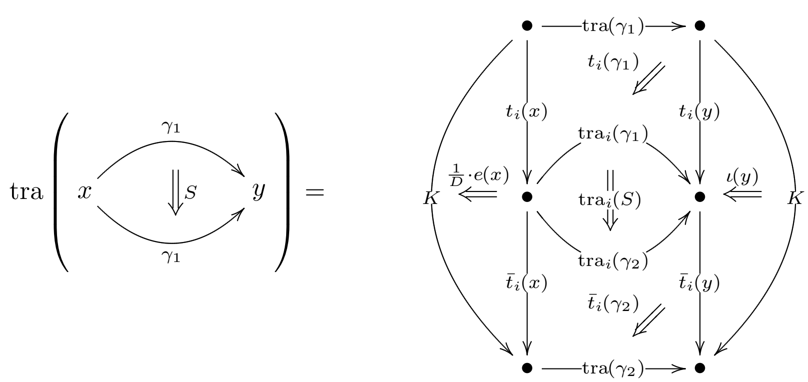
$$S \in \mathrm{Mor}_2\left(\mathcal{P}_2\left(U_i\right)\right)$$

in terms of the local transport



as follows.

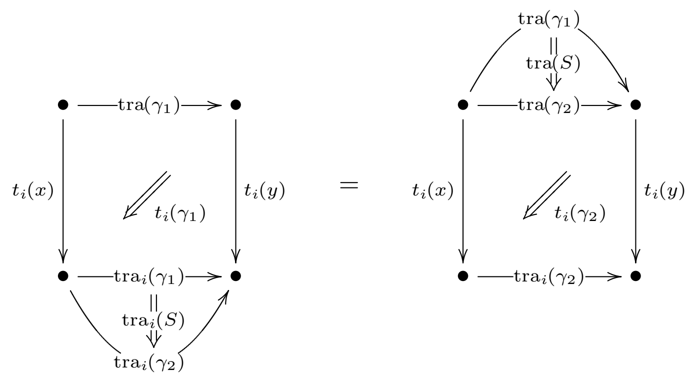
Proposition 2



Proof. Use the tin can equation for the pseudonatural transformation

$$\mathrm{tra}|_{U_i} \xrightarrow{t_i} \mathrm{tra}_i$$

which reads



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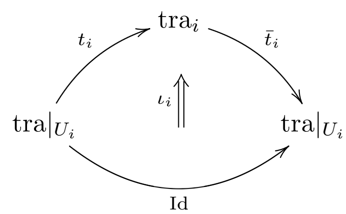
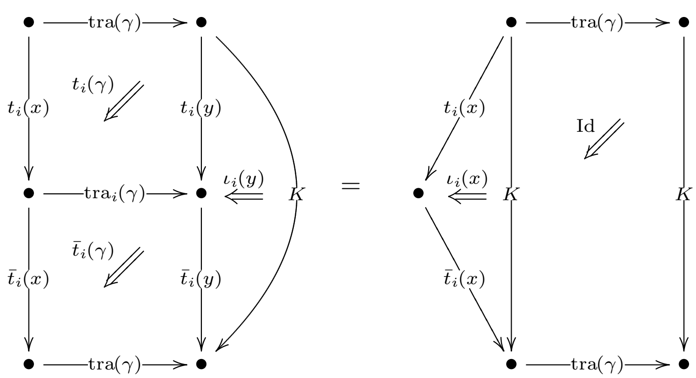
as well as the tin can equation for the modification



,

which reads



□

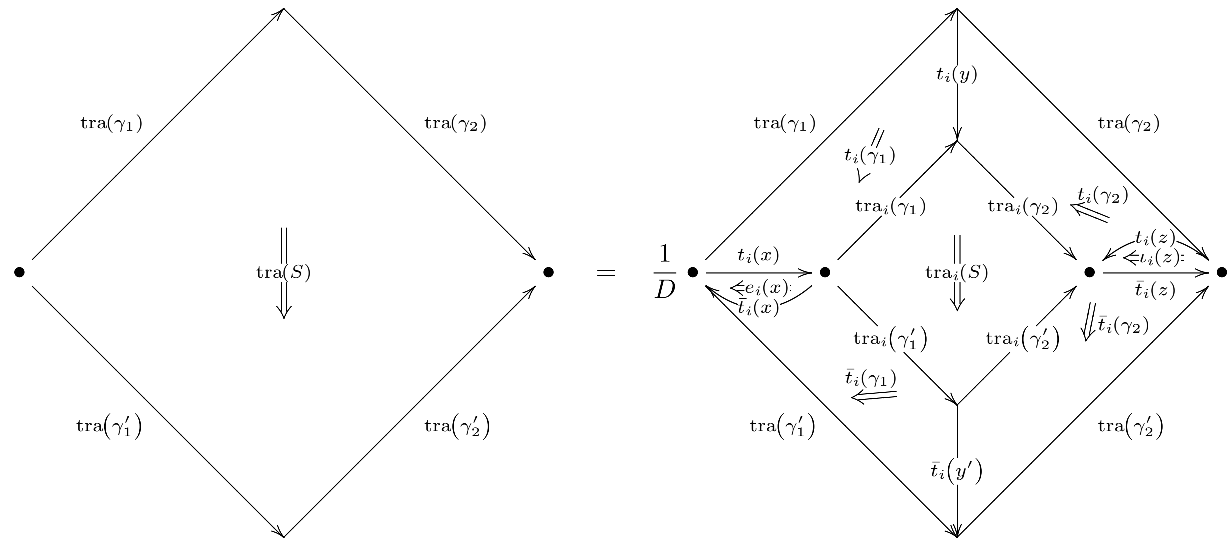
In the following we need to consider surfaces with prescribed decomposition of source

$$x \xrightarrow{\gamma} z \;=\; x \xrightarrow{\gamma_1} y \xrightarrow{\gamma_2} z$$

and target paths

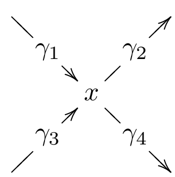
$$x \xrightarrow{\gamma'} z \;=\; x \xrightarrow{\gamma_1'} y' \xrightarrow{\gamma_2'} z \;.$$

By the above proposition, and using the "functoriality" of pseudonatural transformations, their 2-transport can locally be expressed as

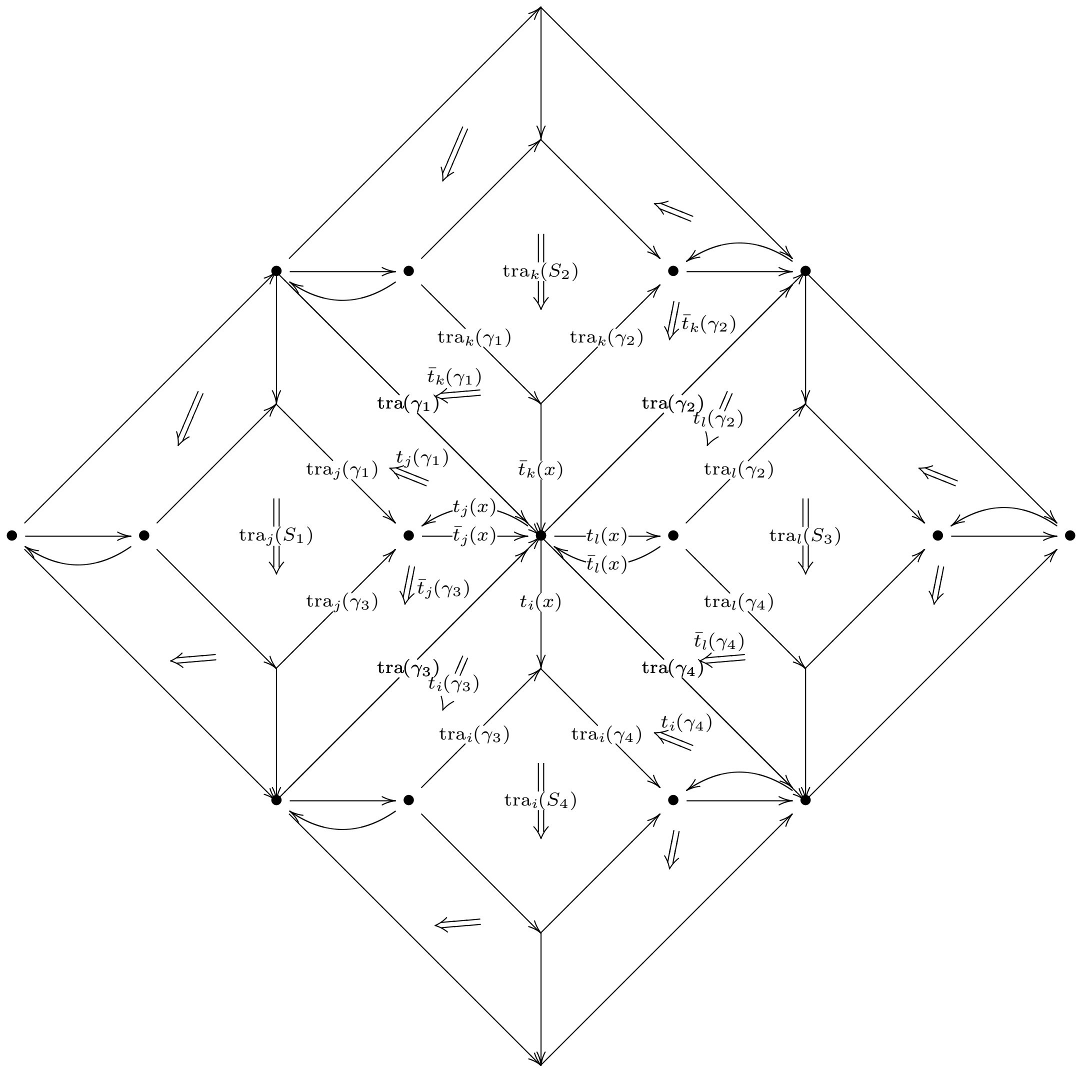


(1)

for $S \in \mathrm{Mor}_2(\mathcal{P}_2(U_i))$. We want to study what happens when four of such surfaces meet in a common point,



like this

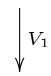


. (2)

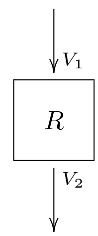
For analyzing this situation it is convenient to adopt a diagrammatic notation *dual* to the one above.

3.1 Dual Graphs

There is a well known powerful diagrammatic calculus for computations in monoidal 1-categories (e.g. section 2.3 of [4] or chaper XIV.1 of [5]). In this context objects are represented by arrows

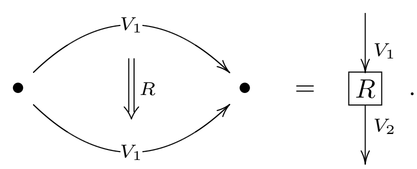


and morphisms $V_1 \xrightarrow{R} V_2$ are represented by boxes



with the source object given by the incoming and the target object given by the outgoing arrow.

This diagrammatic language can be understood as the *dual* of the 2-morphism notation for the same category regarded as a 2-category with a single object:

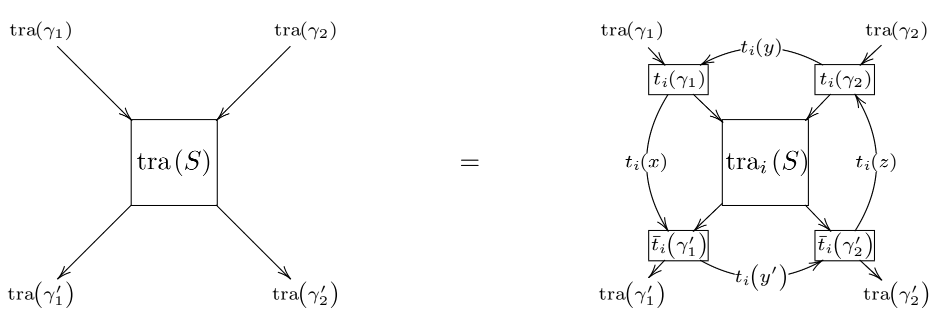


In passing from the 2-categorical 2-morphism description to its dual, one replaces points by surfaces, arrows by perpendicular arrows and surfaces by boxes.

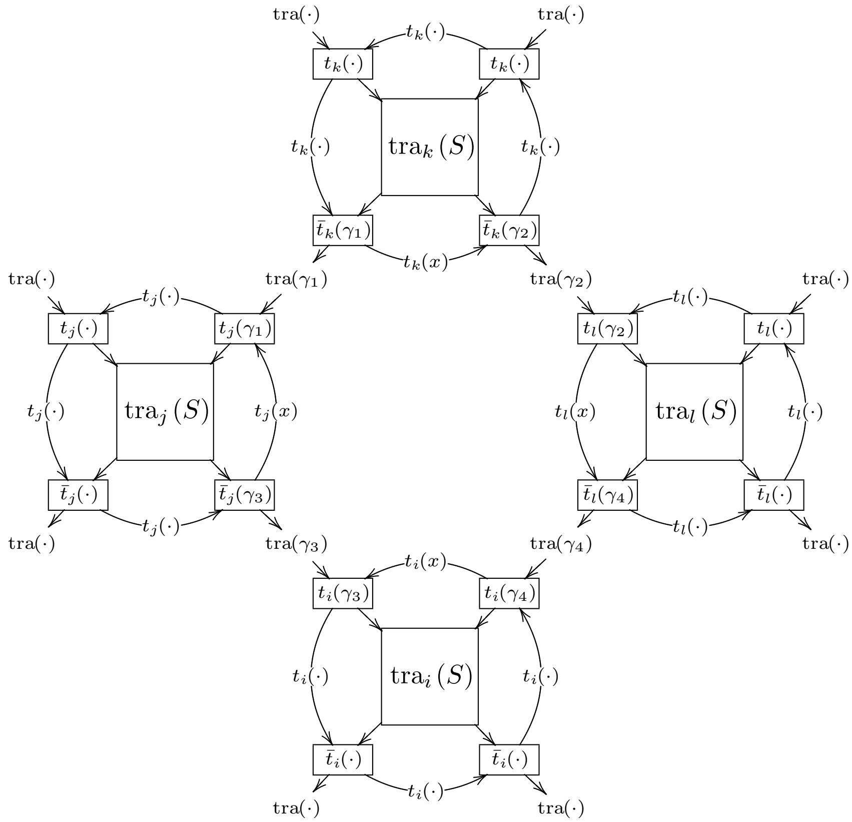
The dual description has the advantage that it automatically takes care of the maps e and ι. [...]

3.2 Surface Transport Computation in Dual Diagrams

In dual notation the local trivialization of the surface transport depicted in (1) looks like



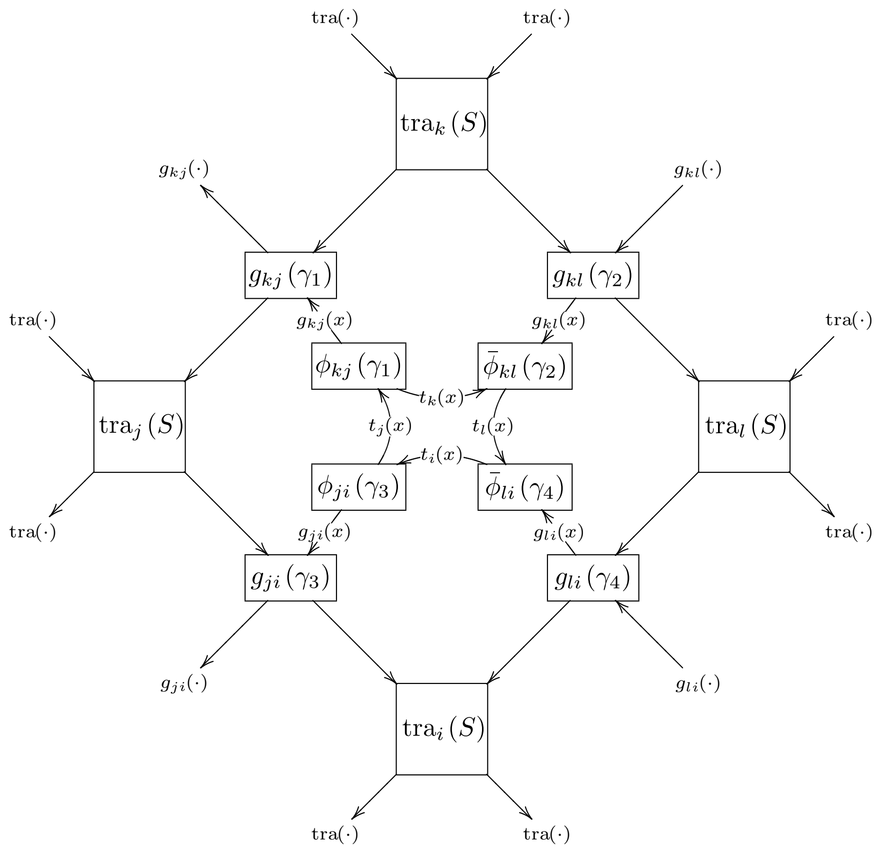
and the composition of four surface elements in (2) becomes



(3)

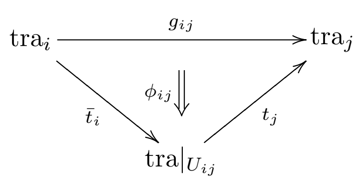
The interior of this dual diagram is easily transformed as follows.

Proposition 3 *The interior of (3) equals*

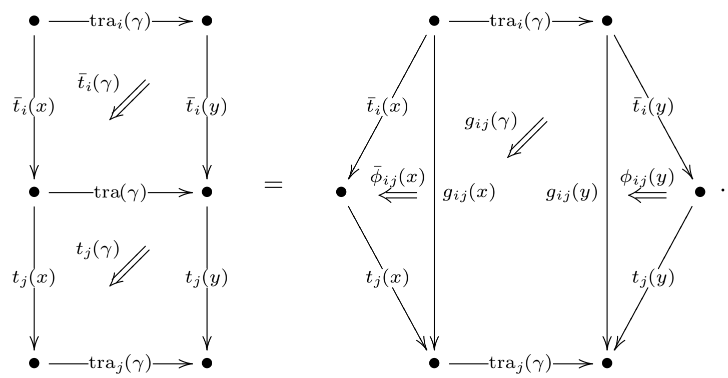


. (4)

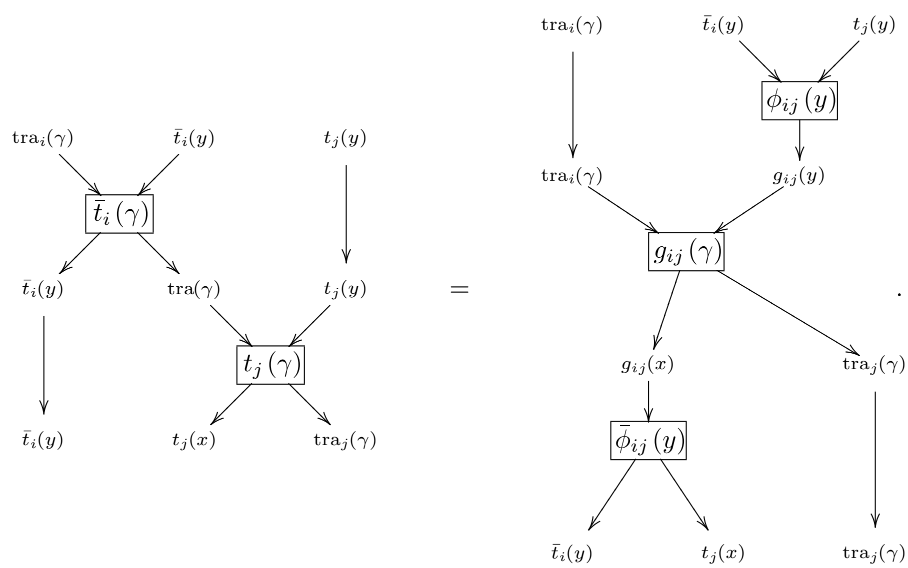
Proof. The tin can equation for the local 2-trivialization modification



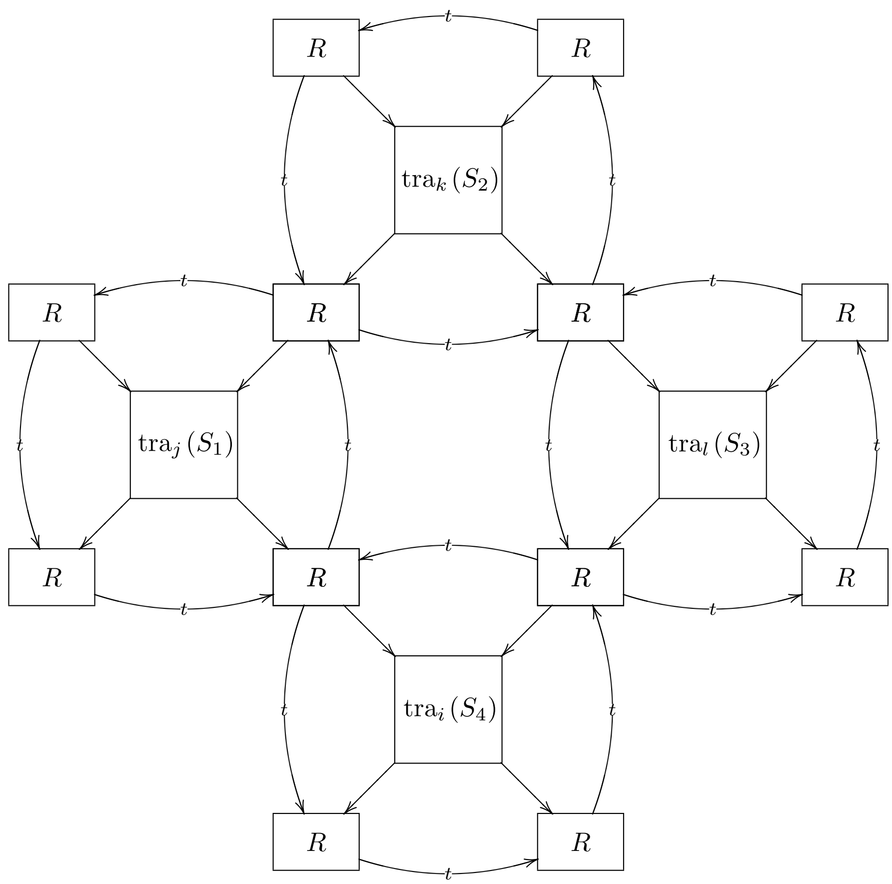
is



The dual of this tin can equation looks like



We can apply this equation at all positions labeled "R" in the following map of diagram (3).



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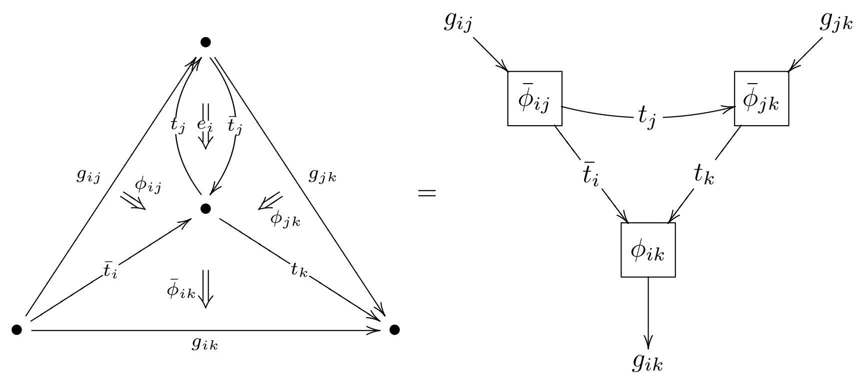
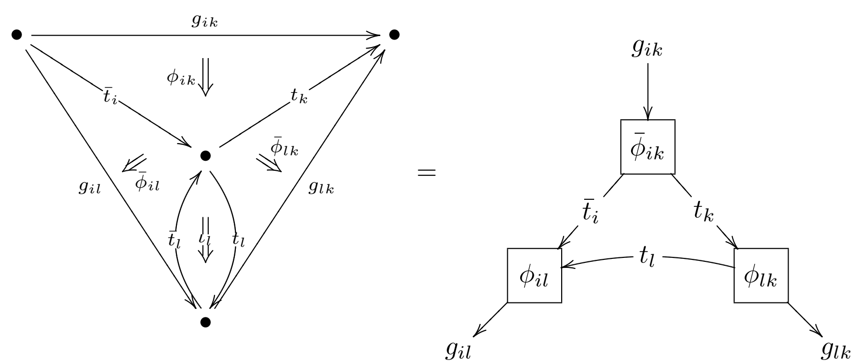
Doing so yields the promised diagram. □

3.3 Transition (Co)-Products

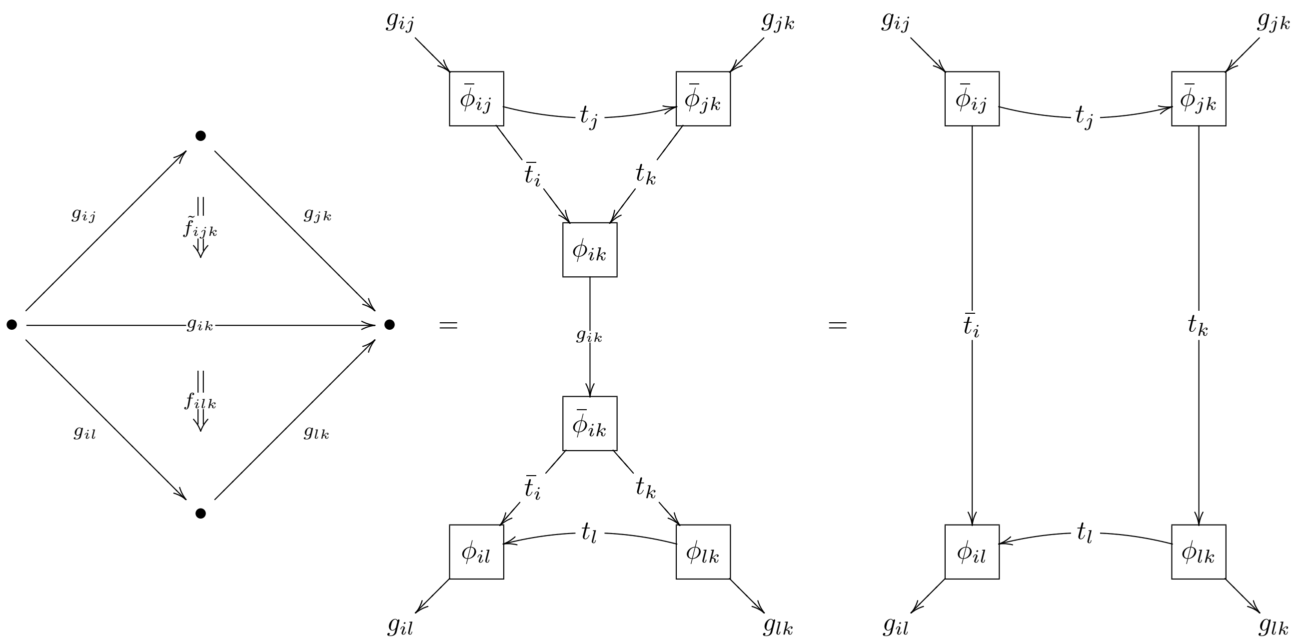
The modifications f_{ijk} and $\tilde{f}_{ijk}$ from def. 2 come with 2-morphism



and



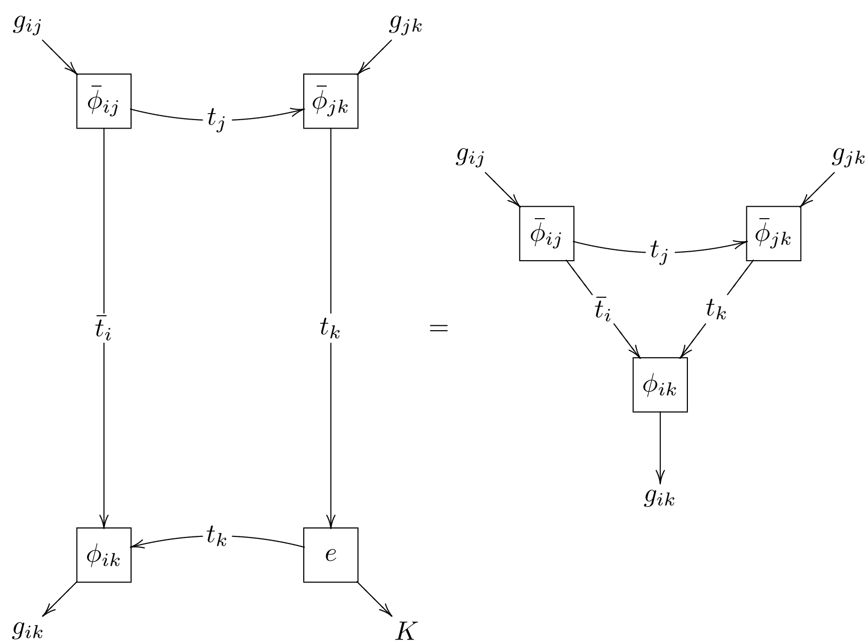
in **Vect**, where all symbols are to be evaluated at some $x \in \mathrm{Obj}(\mathcal{P}_2(U_{ijkl}))$. In terms of the 2-trivializations $\phi_{\cdot\cdot}$ the "Frobenius morphism" reads



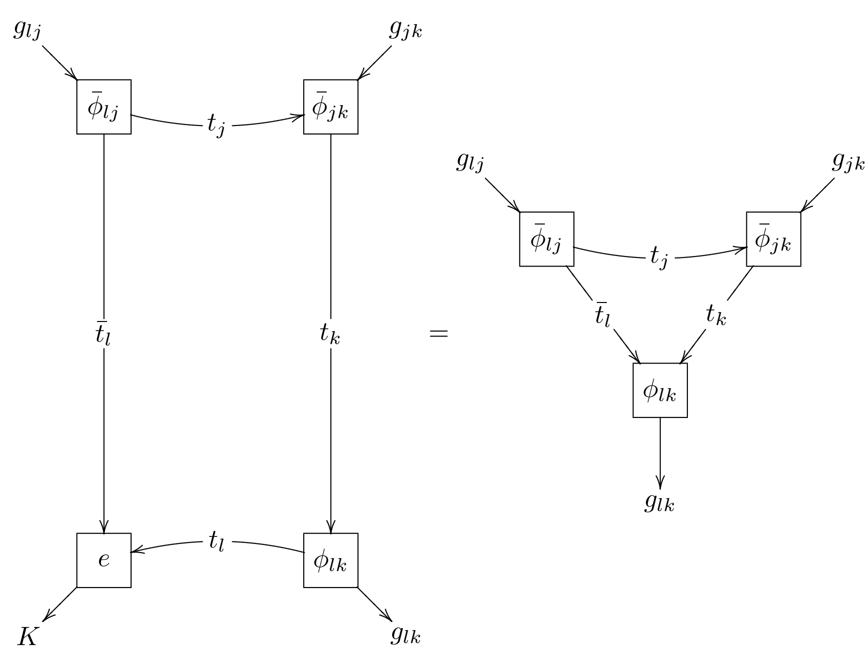
This is precisely the "interior" 2-morphism found in prop. 3, where it arose from composing the locally trivialized 2-transport of four adjacent surface elements.

We are interested in the case where for coinciding indices $\bar{\phi}_{ii} = \iota$ is the unit and $\phi_{ii} = e$ is the counit. In these cases the "Frobenius morphism" reduces to the "product" or "coproduct" whenever two indices coincide:

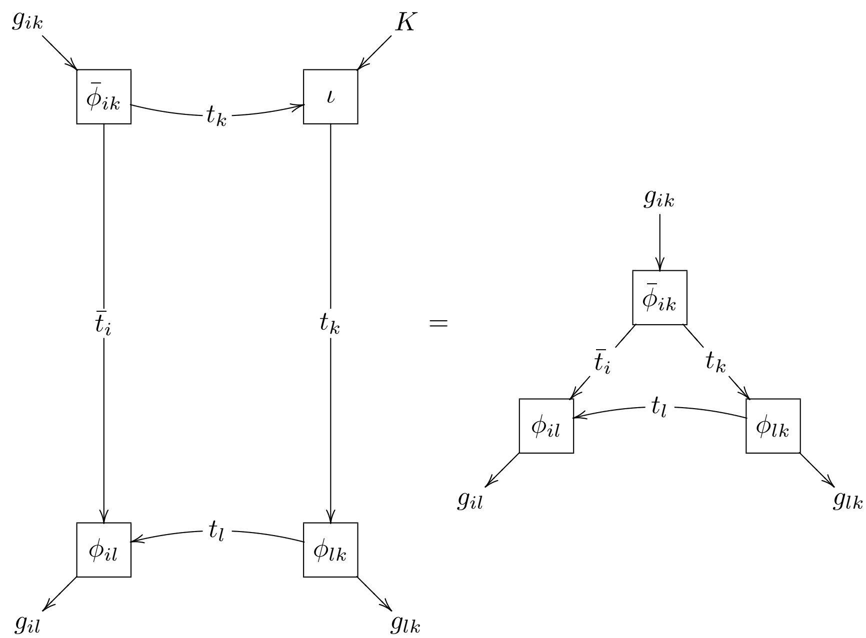
- $\underline{k = l}$:



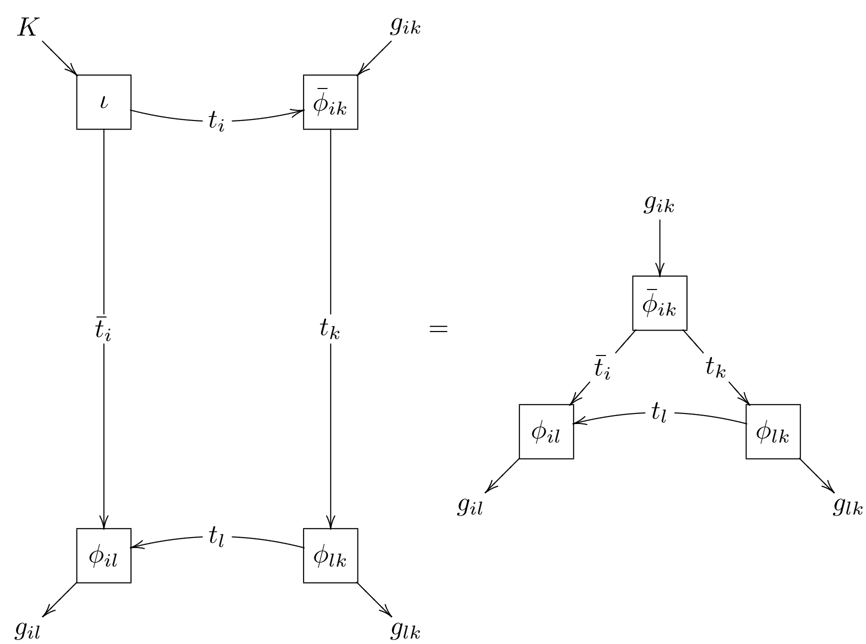
- $\underline{i = l}$:



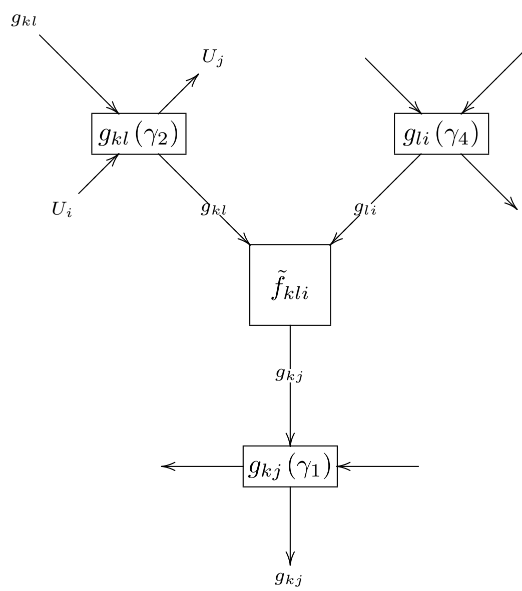
- $\underline{j = k}$:



- $\underline{i = j}$:



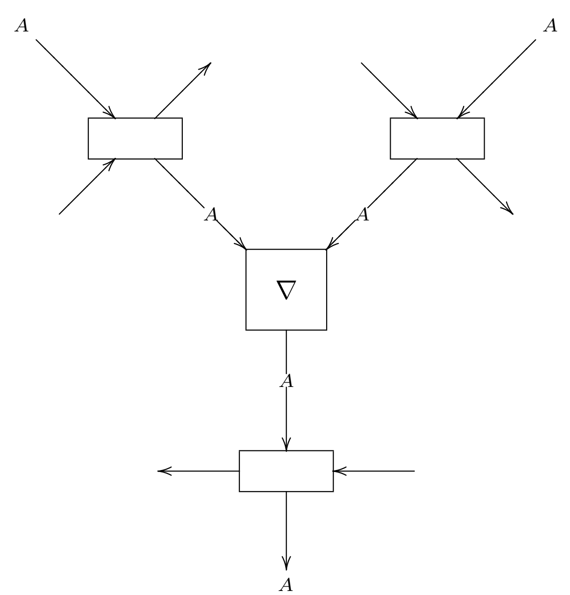
This allows us to obtain the locally trivialized 2-transport diagram for situations where three (instead of four) surface elements meet in a common point. By setting, for instance, $i = j$ in (4), we obtain a dual diagram of the form



(5)

Note that this is the analog in **Vect** of the situation depicted in figure 8 on p. 24 of [11].

Suppose all labeled vector spaces in this diagram happen to coincide, such that we get



Diagrams of this sort appear in the algebraic formulation of TFT/CFT [6, 7].

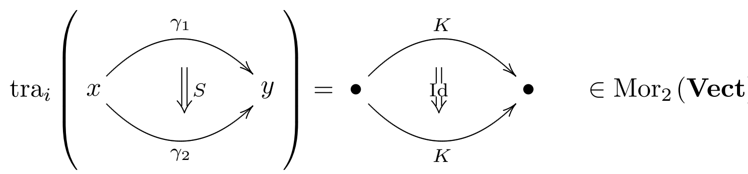
In the next section we identify those transport 2-functors and their local trivializations which give rise to triangulation decorations of this kind.

4 2-Transport with Lattice TFT Holonomy

The results of §3 suggest that there are classes of transport 2-functors whose full local trivializations reproduce the 2-dimensional lattice TFTs introduced in [1]. This is made precise in the following.

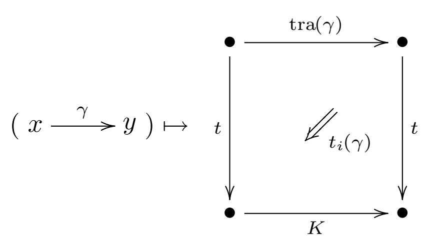
Definition 3 *Let* $\mathrm{tra} : \mathcal{P}_2(M) \to \mathbf{Vect}$ *be a transport 2-functor and let* $T = (\mathcal{U}, \{\mathrm{tra}_i\}, \ldots)$ *be one of its full local trivializations. We say that* T *is* **of FHK-type** *if the following holds*

1. tra *is* **locally flat** *such that for all* $i \in I$ *and all* $S \in \mathrm{Mor}_2(\mathcal{P}_2(U_i))$

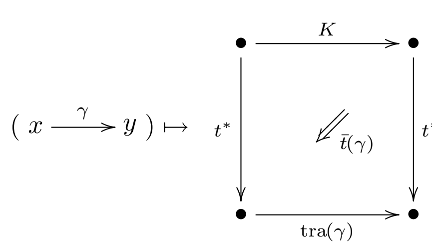


2. **2-transitions are constant** *in the sense that*

   (a) *the trivialization morphism* $\mathrm{tra}|_{U_i} \xrightarrow{t_i} \mathrm{tra}_i$ *is given by*



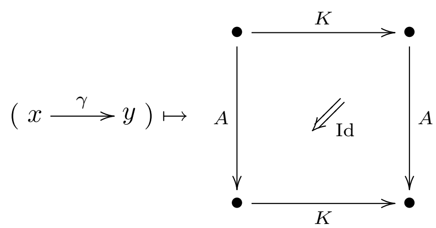
   and $\mathrm{tra}_i \xrightarrow{\bar{t}_i} \mathrm{tra}|_{U_i}$ *is given by*



   with t *some fixed vector space and* t^* *its dual*

   (b) *the transition morphism* $\mathrm{tra}_i \xrightarrow{g_{ij}} \mathrm{tra}_j$ *is the identity morphism for*

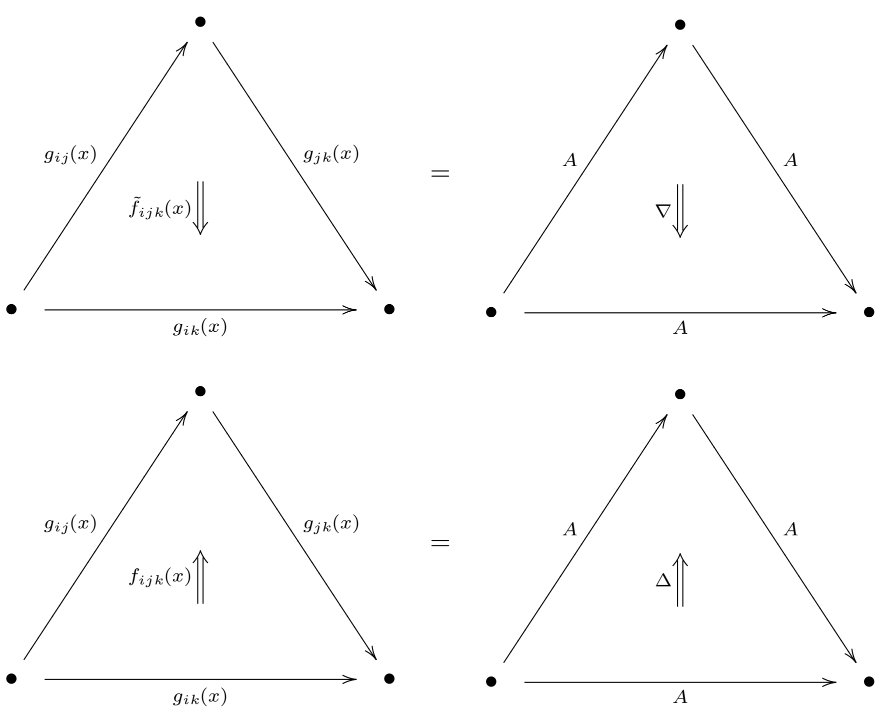
$i = j$ and is otherwise given by



with A some fixed vector space

(c) ϕ_{ii}, $\bar{\phi}_{ii}$ come from the unit and counit on t, respectively, and $\phi_{ij} = \rho$ for all $i \neq j$ with $A \xrightarrow{\rho} t^* \otimes t$ some fixed isomorphism.

Remark. A full 2-transport trivialization of FHK-type defines a vector space A with a fixed isomorphism to the vector space $\mathrm{End}(t)$ of endomorphisms of the vector space t. The transitions



on triple overlaps define the product and coproduct in $\mathrm{End}(t)$. Computing the surface transport of a 2-functor with local trivialization of FHK-type hence reproduces the TFT prescription introduced in [1].

In fact, the TFT defined in [1] require just anz associative semisimple algebra, whereas what we found above is that A is the algebra of endomorphisms of some vector space t. However, **Wedderburn's theorem** states [8] that every associative semisimple algebra is isomorphic to a direct sum of matrix algebras.

5 Outlook

The above discussion pertains only to vacuum amplitudes. Insertions of bulk and boundary fields are naturally associated to the kind of objects labeled U_i and U_j in (5). This does in fact reproduce the known prescription for field insertions in algebraic CFT. Details will be discussed elsewhere.

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