Fachbereich Mathematik | Universität Hamburg
Lecturer: Prof. Mathias Schacht | Fall 2017/18
Assistant: Oliver Ebsen | November 9th, 2017

Graph Theory 2

Exercise Sheet 4

due on November 16, 1pm

http://bit.ly/2hmWDG0

Exercise 1 (§4.29) [1 Punkt]

A family of subgraphs of G is said to form a *double cover* of G if every edge of G lies in exactly two of those subgraphs. A double cover by cycles is a *cycle double cover*.

Let G be a 2-connected graph whose cycle space is generated by a sparse set $\mathscr{C}$ of cycles. From MacLane's theorem we know that G even admits a double cover by cycles generating $\mathcal{C}(G)$: the face boundaries in any drawing of G. Show directly (without using MacLane's theorem) that $\mathscr{C}$ extends to a cycle double cover $\mathscr{D}$ of G.

Exercise 2 (§4.35) [1 Punkt]

Show that a connected plane multigraph has a plane dual.

Exercise 3 (§4.37) [1 Punkt]

Let G^* be an abstract dual of G, and let $e = e^*$ be an edge. Prove the following two assertions:

(*i*) G^*/e^* is an abstract dual of $G - e$.

(*ii*) $G^* - e^*$ is an abstract dual of G/e.

Exercise 4 (§4.42) [1 Punkt]

Show that the following statements are equivalent for connected multigraphs $G = (V, E)$ and $G' = (V', E)$ with the same edge set:

(*i*) G and G' are abstract duals of each other;

(*ii*) given any set $F \subseteq E$, the multigraph (V, F) is a tree if and only if $(V', E \setminus F)$ is a tree.

Written Exercise (§4.28)

Find an algebraic proof of Euler's formula for 2-connected plane graphs, along the following lines. Define the *face space* $\mathcal{F}$ (over $\mathbb{F}_2$) of such a graph in analogy to its vertex space $\mathcal{V}$ and edge space $\mathcal{E}$. Define *boundary maps* $\mathcal{F} \to \mathcal{E} \to \mathcal{V}$ in the obvious way, specifying them first on single faces or edges (i.e., on the standard bases of $\mathcal{F}$ and $\mathcal{E}$) and then extending these maps linearly to all of $\mathcal{F}$ and $\mathcal{E}$. Determine the kernels and images of these homomorphisms, and derive Euler's formula from the dimensions of those subspaces of $\mathcal{F}$, $\mathcal{E}$, and $\mathcal{V}$.

Does such a proof extend to just connected (or even arbitrary) plane graphs?