Fachbereich Mathematik
Diskrete Mathematik
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Probabilistic Method and Random Graphs

8. Series

due on Dezemeber 18

Exercise 1
Let $G = (V, E)$ be a cycle of length kn and let $V = V_1 \dot\cup \ldots \dot\cup V_n$ be a partition of its vertex set into sets of size k.

(i) For which integer k can you show by using the Local Lemma that there exists an independent set of G containing one vertex from each V_i?

(ii) The proof of the Local Lemma yields the following slightly stronger so-called *lopsided local lemma*: Let $A_1, \ldots, A_N$ be events in the same probability space and let $D = ([N], \vec{E})$ be a directed graph. If there exist real numbers $x_1, \ldots, x_N \in [0, 1)$ such that for $i \in [N]$ and every set $S \subseteq [N] \setminus N_D^+(i)$ we have

$$\mathbb{P}\Big(A_i \Big| \bigcap_{s \in S} \bar{A}_s\Big) \leq x_i \prod_{j \in N_D^+(i)} (1 - x_j)\,,$$

then $\mathbb{P}(\bigcap_{i \in [N]} \bar{A}_i) > 0$.

Prove the assertion from (i) for $k = 4$.

Exercise 2
For which p can you show that a.a.s. the random bipartite graph $G(n, n, p)$ (with vertex classes of size n and edges chosen independently with probability p) contains a perfect matching?

Hint: Hall's condition?

Exercise 3
Show that a.a.s. $G(n, 1/2)$ contains no bipartite subgraph with more than $n^2/8 + n^{3/2}$ edges.