Fachbereich Mathematik
Diskrete Mathematik
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Probabilistic Method and Random Graphs

3. Series

due on November 6

Exercise 1
Szele's theorem asserts that there are n-vertex tournaments containing at least $n!/2^{n-1}$ Hamiltonian paths. Show that every tournament contains at least one Hamiltonian path.
Hint: No probability needed here.

Exercise 2
Suppose k persons enter the same lift at the ground level and there are n possible stops (excluding the current stop at ground floor). What is the expected number of stops until everybody left the lift? (Here we assume that nobody else stops the lift on the way and the "exit floors" are uniformly distributed for each person, i.e., $\mathbb{P}(\text{person } a \text{ wants to leave at level } i) = 1/n$ for every $i = 1, \dots, n$.

Exercise 3
Let Ω be a finite and non-empty set. Consider the uniform sample space on Ω, i.e., we consider the probability space $(\Omega, 2^{\Omega}, \mathbb{P})$ given by $\mathbb{P}(\{\omega\}) = |\Omega|^{-1}$. Let $X\colon\ \Omega \to \{0, \dots, M\}$ be a random variable with expectation $\mathbb{E}X = M - a$ for some $a \in \mathbb{R}$. Prove that for every $b \in \mathbb{R}$ with $1 \le b \le M$ we have

$$\mathbb{P}(X \ge M - b) \ge \frac{b-a}{b}\,.$$

Exercise 4
Prove Turán's theorem from extremal graph theory in the complementary form: *If an n-vertex graph G has average degree k, then $\alpha(G) \ge \frac{n}{k+1}$.*

(i) Show that for any graph $G = (V, E)$ we have

$$\alpha(G) \ge \sum_{v \in V} \frac{1}{d(v)+1}\,.$$

Hint: Consider a random ordering of V and the set of vertices with the property that all their neighbours are behind them in the ordering.

(ii) Deduce Turán's theorem from part (i).

Hint: Cauchy-Schwarz inequality