Probabilistic Method and Random Graphs

2. Series

due on October 30

Exercise 1
Prove a lower bound for the symmetric Ramsey number $R^{(k)}(n,n)$ for k-uniform hypergraphs.

Exercise 2
A *tournament* is a directed graph without loops with precisely one directed arc for every pair of vertices. We say a tournament $T=(V,\vec{E})$ has the property $\mathcal{P}(k,\ell)$ for integers k, $\ell\in\mathbb{N}$, if $|V|\geq k+\ell$ and for any $K\in\binom{V}{k}$ there exists an $L\in\binom{V\setminus K}{\ell}$ such that $K\times L\subseteq\vec{E}$. In other words, T has $\mathcal{P}(k,\ell)$ if for any set of k vertices there are at least ℓ vertices which "beat" them.

(i) Show that there exists a tournament on $C(k\ell)^2 2^{k\ell}$ vertices with the property $\mathcal{P}(k,\ell)$ for some constant C independent of k and ℓ.

Hint: Consider the case $\ell=1$ first.

(ii) Show that every tournament satisfying $\mathcal{P}(k,1)$ must have at least $c2^k$ vertices for some constant $c>0$ independent of k and ℓ.

(iii) Try to improve the lower bound given in (ii) to $ck2^k$.

Hint: Consider a lower bound for $\mathcal{P}(k-1,k+1)$ for this.

Exercise 3
A k-uniform hypergraph is r-*colourable*, if there exists a partition into at most r classes such that no hyperedge is contained in one of the classes. Show for k, $r\geq 2$ that if a k-uniform hypergraph $H=(V,E)$ satisfies $|E|<r^{k-1}$, then H is r-colourable. Is the same true if $|E|=r^{k-1}$?