Loose Hamiltonian cycles forced by large $(k-2)$-degree - sharp version

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Abstract

We prove for all $k \geqslant 4$ and $1 \leqslant \ell < k/2$ the sharp minimum $(k-2)$-degree bound for a k-uniform hypergraph $\mathcal{H}$ on n vertices to contain a Hamiltonian ℓ-cycle if $k-\ell$ divides n and n is sufficiently large. This extends a result of Han and Zhao for 3-uniform hypergraphs.

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1 Introduction

Given $k \geqslant 2$, a k-uniform hypergraph $\mathcal{H}$ is a pair (V, E) with vertex set V and edge set $E \subseteq V^{(k)}$ being a subset of all k-element subsets of V. Given a k-uniform hypergraph $\mathcal{H} = (V, E)$ and a subset $S \in V^{(s)}$, we denote by $d(S)$ the number of edges in E containing S and we denote by $N(S)$ the $(k-s)$-element sets $T \in V^{(k-s)}$ such that $T \cup S \in E$, so $d(S) = |N(S)|$. The *minimum s-degree* of $\mathcal{H}$ is denoted by $\delta_s(\mathcal{H})$ and it is defined as the minimum of $d(S)$ over all sets $S \in V^{(s)}$.

We say that a k-uniform hypergraph $\mathcal{C}$ is an *ℓ-cycle* if there exists a cyclic ordering of its vertices such that every edge of $\mathcal{C}$ is composed of k consecutive vertices, two (vertex-wise) consecutive edges share exactly ℓ vertices, and every vertex is contained in an edge. Moreover, if the ordering is not cyclic, then $\mathcal{C}$ is an *ℓ-path* and we say that the first and last ℓ vertices are the ends of the path. The problem of finding minimum degree conditions that ensure the existence of Hamiltonian cycles, i.e. cycles that contain all vertices of a given hypergraph, has been extensively studied over the last years (see, e.g., the surveys [11,14]). Katona and Kierstead [7] started the study of this problem, posing a conjecture that was confirmed by Rödl, Ruciński, and Szemerédi [12,13], who proved the following result: For every $k \geqslant 3$, if $\mathcal{H}$ is a k-uniform n-vertex hypergraph with $\delta_{k-1}(\mathcal{H}) \geqslant (1/2 + o(1))n$, then $\mathcal{H}$ contains a Hamiltonian $(k-1)$-cycle. Kühn and Osthus proved that 3-uniform hypergraphs $\mathcal{H}$ with $\delta_2(\mathcal{H}) \geqslant (1/4 + o(1))n$ contain a Hamiltonian 1-cycle [10], and Hàn and Schacht [4] (see also [8]) generalized this result to arbitrary k and ℓ-cycles with $1 \leqslant \ell < k/2$. In [9], Kühn, Mycroft, and Osthus generalized this result to $1 \leqslant \ell < k$, settling the problem of the existence of Hamiltonian ℓ-cycles in k-uniform hypergraphs with large minimum $(k-1)$-degree. In Theorem 1.1 below (see [1,3]) we have minimum $(k-2)$-degree conditions that ensure the existence of Hamiltonian ℓ-cycles for $1 \leqslant \ell < k/2$.

Theorem 1.1 *For all integers $k \geqslant 3$ and $1 \leqslant \ell < k/2$ and every $\gamma > 0$ there exists an n_0 such that every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with $n \in (k-\ell)\mathbb{N}$ and*

$$\delta_{k-2}(\mathcal{H}) \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} + \gamma\right)\binom{n}{2}$$

contains a Hamiltonian ℓ-cycle. □

The minimum degree condition in Theorem 1.1 is asymptotically optimal as the following well-known example confirms. The construction of the example varies slightly depending on whether n is an odd or an even multiple of $k-\ell$. We first consider the case that $n=(2m+1)(k-\ell)$ for some integer m. Let $\mathcal{X}_{k,\ell}(n)=(V,E)$ be a k-uniform hypergraph on n vertices such that an edge belongs to E if and only if it contains at least one vertex from $A\subset V$, where $|A|=\left\lfloor\frac{n}{2(k-\ell)}\right\rfloor$. It is easy to see that $\mathcal{X}_{k,\ell}(n)$ contains no Hamiltonian ℓ-cycle, as it would have to contain $\frac{n}{k-\ell}$ edges and each vertex in A is contained in at most two of them. Indeed any maximal ℓ-cycle includes all but $k-\ell$ vertices and adding any additional edge to the hypergraph would imply a Hamiltonian ℓ-cycle. Let us now consider the case that $n=2m(k-\ell)$ for some integer m. Similarly, let $\mathcal{X}_{k,\ell}(n)=(V,E)$ be a k-uniform hypergraph on n vertices that contains all edges incident to $A\subset V$, where $|A|=\frac{n}{2(k-\ell)}-1$. Additionally, fix some $\ell+1$ vertices of $B=V\smallsetminus A$ and let $\mathcal{X}_{k,\ell}(n)$ contain all edges on B that contain all of these vertices, i.e., an $(\ell+1)$-star. Again, of the $\frac{n}{k-\ell}$ edges that a Hamiltonian ℓ-cycle would have to contain, at most $\frac{n}{k-\ell}-2$ can be incident to A. So two edges would have to be completely contained in B and be disjoint or intersect in exactly ℓ vertices, which is impossible since the induced subhypergraph on B only contains an $(\ell+1)$-star. Note that for the minimum $(k-2)$-degree the $(\ell+1)$-star on B is only relevant if $\ell=1$, in which case this star increases the minimum $(k-2)$-degree by one. In Theorem 1.2 below we obtain the optimal bound for $\delta_{k-2}(\mathcal{H})$ for any $k\geqslant 4$.

Theorem 1.2 *For all integers $k\geqslant 4$ and $1\leqslant\ell<k/2$ there exists n_0 such that every k-uniform hypergraph $\mathcal{H}=(V,E)$ on $|V|=n\geqslant n_0$ vertices with $n\in(k-\ell)\mathbb{N}$ and*

$$\delta_{k-2}(\mathcal{H})>\delta_{k-2}(\mathcal{X}_{k,\ell}(n)) \tag{1}$$

contains a Hamiltonian ℓ-cycle. In particular, if

$$\delta_{k-2}(\mathcal{H})\geqslant\frac{4(k-\ell)-1}{4(k-\ell)^2}\binom{n}{2},$$

then $\mathcal{H}$ contains a Hamiltonian ℓ-cycle.

The following notion of extremality is motivated by the hypergraph $\mathcal{X}_{k,\ell}(n)$. A k-uniform hypergraph $\mathcal{H}=(V,E)$ is called (ℓ,ξ)-*extremal* if there exists a partition $V=A\cup B$ such that $|A|=\left\lceil\frac{n}{2(k-\ell)}-1\right\rceil$, $|B|=\left\lfloor\frac{2(k-\ell)-1}{2(k-\ell)}n+1\right\rfloor$ and $e(B)=|E\cap B^{(k)}|\leqslant\xi\binom{n}{k}$. We say that $A\cup B$ is an (ℓ,ξ)-*extremal partition* of V. Theorem 1.2 follows easily from the next two results, the so-called *extremal case* (see Theorem 1.3 below) and *non-extremal case* (see Theorem 1.4, which was addressed in [1]).

Theorem 1.3 *For any integers $k \geqslant 3$ and $1 \leqslant \ell < k/2$, there exists $\xi > 0$ such that the following holds for sufficiently large n. Suppose $\mathcal{H}$ is a k-uniform hypergraph on n vertices with $n \in (k-\ell)\mathbb{N}$ such that $\mathcal{H}$ is (ℓ, ξ)-extremal and*

$$\delta_{k-2}(\mathcal{H}) > \delta_{k-2}(\mathcal{X}_{k,\ell}(n)).$$

Then $\mathcal{H}$ contains a Hamiltonian ℓ-cycle.

Theorem 1.4 *For any $0 < \xi < 1$ and all integers $k \geqslant 4$ and $1 \leqslant \ell < k/2$, there exists $\gamma > 0$ such that the following holds for sufficiently large n. Suppose $\mathcal{H}$ is a k-uniform hypergraph on n vertices with $n \in (k-\ell)\mathbb{N}$ such that $\mathcal{H}$ is not (ℓ, ξ)-extremal and*

$$\delta_{k-2}(\mathcal{H}) \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} - \gamma\right)\binom{n}{2}.$$

Then $\mathcal{H}$ contains a Hamiltonian ℓ-cycle. □

In Section 2 we give an overview of the proof of Theorem 1.3.

2 Overview

Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph and let $X, Y \subset V$ be disjoint subsets. Given a vertex set $L \subset V$ we denote by $d(L, X^{(i)}Y^{(j)})$ the number of edges of the form $L \cup I \cup J$, where $I \in X^{(i)}$, $J \in Y^{(j)}$, and $|L| + i + j = k$. We allow for $Y^{(j)}$ to be omitted when j is zero and write $d(v, X^{(i)}Y^{(j)})$ for $d(\{v\}, X^{(i)}Y^{(j)})$.

Let $\varrho > 0$ and integers $k \geqslant 3$ and $1 \leqslant \ell < k/2$ be given and fix small ε and $\xi \ll \varepsilon$. The proof of Theorem 1.3 follows ideas from [5], where a corresponding result with a $(k-1)$-degree condition is proved. Let $n \in (k-\ell)\mathbb{N}$ be sufficiently large and let $\mathcal{H}$ be an (ℓ, ξ)-extremal k-uniform hypergraph on n vertices that satisfies the $(k-2)$-degree condition

$$\delta_{k-2}(\mathcal{H}) > \delta_{k-2}(\mathcal{X}_{k,\ell}(n)).$$

Let $A \,\dot\cup\, B = V(\mathcal{H})$ be a minimal extremal partition of $V(\mathcal{H})$, i.e. a partition satisfying

$$a = |A| = \left\lceil \frac{n}{2(k-\ell)} - 1 \right\rceil, \quad b = |B| = n - a, \quad \text{and} \quad e(B) \leqslant \xi \binom{n}{k},$$

which minimises $e(B)$. We first construct an ℓ-path $\mathcal{Q}$ in $\mathcal{H}$ with ends L_0 and L_1 such that there is a partition $A_* \,\dot\cup\, B_*$ of $(V(\mathcal{H}) \smallsetminus V(\mathcal{Q})) \cup L_0 \cup L_1$ composed only of "typical" vertices (see *(ii)* and *(iii)* below). The set $A_* \cup B_*$ is suitable for an application of Lemma 3.10 from [5], which ensures the existence of an ℓ-path $\mathcal{Q}'$ on $A_* \cup B_*$ with L_0 and L_1 as ends. Note that the existence of a Hamiltonian ℓ-cycle in $\mathcal{H}$ is guaranteed by $\mathcal{Q}$ and $\mathcal{Q}'$. So, in order to prove Theorem 1.3, we only need to prove the following.

(i) $|B_*| = (2k - 2\ell - 1)|A_*| + \ell$,

(ii) $d(v, B_*^{(k-1)}) \geqslant (1 - \varrho)\binom{|B_*|}{k-1}$ for any vertex $v \in A_*$,

(iii) $d(v, A_*^{(1)} B_*^{(k-2)}) \geqslant (1 - \varrho)|A_*|\binom{|B_*|}{k-2}$ for any vertex $v \in B_*$,

(iv) $d(L_0, A_*^{(1)} B_*^{(k-\ell-1)}), d(L_1, A_*^{(1)} B_*^{(k-\ell-1)}) \geqslant (1 - \varrho)|A_*|\binom{|B_*|}{k-\ell-1}$.

Since $e(B) \leqslant \xi\binom{n}{k}$, we expect most vertices $v \in B$ to have low degree $d(v, B^{(k-1)})$ into B. Also, most $v \in A$ must have high degree $d(v, B^{(k-1)})$ into B such that the degree condition for $(k-2)$-sets in B can be satisfied. Thus, we define the sets A_ε and B_ε to consist of vertices of high respectively low degree into B by

$$\begin{aligned} A_\varepsilon &= \left\{ v \in V : d(v, B^{(k-1)}) \geqslant (1 - \varepsilon)\binom{|B|}{k-1} \right\}, \\ B_\varepsilon &= \left\{ v \in V : d(v, B^{(k-1)}) \leqslant \varepsilon\binom{|B|}{k-1} \right\}, \end{aligned}$$

and put $V_\varepsilon = V \smallsetminus (A_\varepsilon \cup B_\varepsilon)$. It follows from these definitions that

$$\text{if } A \cap B_\varepsilon \neq \varnothing, \quad \text{then} \quad B \subset B_\varepsilon, \quad \text{otherwise} \quad A \subset A_\varepsilon. \tag{2}$$

Actually, we can show that the sets A_ε and B_ε are not too different from A and B respectively and that V_ε is small. Since we are interested in ℓ-cycles, the degree of ℓ-tuples in B_ε will be of interest, which motivates the following definition. An ℓ-set $L \subset B_\varepsilon$ is called ε-*typical* if $d(L, B^{(k-\ell)}) \leqslant \varepsilon\binom{|B|}{k-\ell}$. Indeed, most ℓ-sets in B_ε are ε-typical and ε-typical sets can be connected using ℓ-paths of size two in a robust way, i.e., avoiding any small set of vertices.

We want to construct an ℓ-path $\mathcal{Q}$ with ends L_0 and L_1, such that $V_\varepsilon \subset V(\mathcal{Q})$ and the remaining sets $A_* = A_\varepsilon \smallsetminus V(\mathcal{Q})$ and $B_* = (B_\varepsilon \smallsetminus V(\mathcal{Q})) \cup L_0 \cup L_1$ have the right proportion of vertices, i.e., one to $(2k - 2\ell - 1)$. We can find $|V_\varepsilon|$ paths of size two with ε-typical ends that each contain a distinct vertex of V_ε and otherwise contain vertices from B_ε. If $|A \cap B_\varepsilon| > 0$, then $B \subset B_\varepsilon$ and so $\mathcal{Q}$ should cover V_ε and contain the right number of vertices from B_ε. For this, we can find $2|A \cap B_\varepsilon|$ disjoint paths of size three, each of which contains exactly one vertex from A_ε and has two ε-typical sets as its ends. If on the other hand A_ε overlaps into B, we can extend one of paths with single edges using exactly one vertex from A_ε to obtain the right proportion.

In both cases, we can connect all the short ℓ-paths as the ends are ε-typical, again keeping the right proportion of vertices in A_ε and B_ε, i.e., satisfying *(i)*. Since the constructed path $\mathcal{Q}$ is small and the difference between A_ε and A as well as B_ε and B is small, the remaining required properties *(ii)*-*(iv)* follow from the definition of A_ε and B_ε.

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