A counting lemma for sparse pseudorandom hypergraphs ⋆

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Abstract

Our main result tells us that mild density and pseudorandom conditions allow one to prove certain counting lemmas for a restricted class of subhypergraphs in a sparse setting. As an application, we present a variant of a universality result of Rödl for sparse, 3-uniform hypergraphs contained in strongly pseudorandom hypergraphs.

Keywords: Embeddings, hypergraphs, pseudorandomness

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1 Introduction and main results

We say that a graph $G = (V, E)$ satisfies property $\mathcal{Q}(\eta, \delta, \alpha)$ if, for every subgraph $G[S]$ induced by $S \subset V$ such that $|S| \geq \eta|V|$, we have $(\alpha - \delta)\binom{|S|}{2} < |E(G[S])| < (\alpha + \delta)\binom{|S|}{2}$. In [10], answering affirmatively a question posed by Erdős (see, e.g., [1] and [5]), Rödl proved that for every positive integer m and for every positive $\alpha, \eta < 1$ there exist $\delta > 0$ and an integer $n_0 > 0$ such that, if $n \geq n_0$, then every n-vertex graph G satisfying $\mathcal{Q}(\eta, \delta, \alpha)$ contains all graphs with m vertices as induced subgraphs. Note that η is not required to be small in this result, e.g., it could be, say, $1/2$. It is remarkable that uniform edge distribution over such 'large' sets suffices in Rödl's theorem. We prove a variant of this result, which allows one to count the number of embeddings (not necessarily induced labeled copies) of some fixed 3-uniform hypergraphs into spanning subgraphs of "jumbled" 3-uniform hypergraphs.

Before we state our main results, we need some definitions. First, we generalize property $\mathcal{Q}(\eta, \delta, \alpha)$ to 3-uniform hypergraphs. We say that a 3-uniform hypergraph $G = (V, E)$ satisfies property $\mathcal{Q}'(\eta, \delta, q)$ if, for all $X \subset \binom{V}{2}$ and $Y \subset V$ with $|X| \geq \eta\binom{|V|}{2}$ and $|Y| \geq \eta|V|$, we have $(1-\delta)q|X||Y| \leq |E_G(X, Y)| \leq (1+\delta)q|X||Y|$, where $E_G(X, Y)$ denotes the set of edges of G containing a member of X and a member of Y.

A 3-uniform hypergraph $\Gamma = (V, E)$ is called (p, β)-*jumbled* if, for all subsets $X \subset \binom{V}{2}$ and $Y \subset V$, we have $\big||E_\Gamma(X, Y)| - p|X||Y|\big| \leq \beta\sqrt{|X||Y|}$. A k-uniform hypergraph H is called *linear* if every pair of edges shares at most one vertex. An edge e of a linear k-uniform hypergraph $E(H)$ is a *connector* if there exist $v \in V(H) \setminus \{e\}$ and k edges $e_1, \ldots, e_k$ containing v such that $|e \cap e_i| = 1$ for $1 \leq i \leq k$. Note that, for $k = 2$, a connector is an edge that is contained in a triangle.

Finally, we say that a k-uniform hypergraph G satisfies property $\mathrm{BDD}(C, t, p)$ if, for all $1 \leq r \leq t$ and for all distinct $S_1, \ldots, S_r \in \binom{V(G)}{k-1}$, we have $|N_G(S_1) \cap \ldots \cap N_G(S_r)| \leq Cnp^r$.

We estimate the number of copies of small linear, connector-free 3-uniform hypergraphs H contained in n-vertex 3-uniform spanning subhypergraphs G_n of $(p, \gamma p^2 n^{3/2})$-jumbled hypergraphs, for sufficiently small $\gamma > 0$ and sufficiently large p and n. We remark that, if $p \gg n^{-1/4}$, then the random 3-uniform hypergraph, where each possible edge exists with probability p independently of all other edges, is $(p, \gamma p^2 n^{3/2})$-jumbled with high probability, for all $\gamma > 0$. One of our main results is the following theorem. We denote the family of embeddings of H into G_n by $\mathcal{E}(H, G_n)$.

Theorem 1.1 *For all $\varepsilon, \alpha, \eta > 0$, $C > 1$, and an integer $m \geq 4$, there exist $\delta'', \gamma, D > 0$ such that if $p = p(n) \geq Dn^{-1/m}$ with $p = p(n) = o(1)$ and n is sufficiently large, then the following holds for every $\alpha p \leq q \leq p$. Suppose that*

(i) Γ is an n-vertex (p, β)-jumbled 3-uniform hypergraph;

(ii) G_n is a spanning subhypergraph of Γ with $|E(G_n)| = q\binom{n}{3}$ and G_n satisfies $\mathcal{Q}'(\eta, \delta'', q)$ and $\mathrm{BDD}(C, m, q)$.

If $\beta \leq \gamma p^2 n^{3/2}$, then for every linear 3-uniform connector-free hypergraph H on m vertices we have

$$\big||\mathcal{E}(H, G_n)| - n^m q^{e(H)}\big| < \varepsilon n^m q^{e(H)}.$$

Part of the proof of Theorem 1.1 is based on a counting result for small linear, connector-free 3-uniform hypergraphs into n-vertex "pseudorandom" hypergraphs. We say that a k-uniform hypergraph G satisfies property $\mathrm{TUPLE}(t, \delta, p)$ if, for all $1 \leq r \leq t$, we have $\big||N_G(S_1) \cap \ldots \cap N_G(S_r)| - np^r\big| < \delta np^r$ for all but at most $\delta\binom{\binom{n}{k-1}}{r}$ distinct sets $S_1, \ldots, S_r \in \binom{V(G)}{k-1}$. If a k-uniform hypergraph G satisfies properties $\mathrm{BDD}(C, t_1, q)$ and $\mathrm{TUPLE}(t_2, \delta, q)$, and $|E(G)| = q\binom{n}{k}$, then we say that G is *(C, t_1, t_2, δ, q)-pseudorandom*. We remark that similar notions of pseudorandomness in hypergraphs were considered in [6,7].

Given a k-uniform hypergraph H, let $d_H = \max\{\delta(J) \colon J \subset H\}$ and $D_H = \min\{k \cdot d_H, \Delta(H)\}$. The next result, which is our second main theorem, is a generalization for k-uniform hypergraphs of a counting result for graphs proved in [9]. For related results, the reader is referred to [3] and [4].

Theorem 1.2 *Let $k \geq 2$ and $m \geq 4$ be integers. Let H be a k-uniform hypergraph on m vertices and let G_n be an n-vertex k-uniform hypergraph. For all $\varepsilon > 0$ and $C > 1$, there exist $\delta, D > 0$ for which the following holds when $q \geq Dn^{-1/D_H}$ and n is sufficiently large.*

If G_n is $(C, D_H, 2, \delta, q)$-pseudorandom and H is linear and connector-free, then

$$\big||\mathcal{E}(H, G_n)| - n^m q^{e(H)}\big| < \varepsilon n^m q^{e(H)}.$$

The first part of the proof of Theorem 1.1 involves proving that, if a graph G_n is as in the statement of the theorem, then property $\mathcal{Q}'(\eta, \delta'', q)$ implies $\mathrm{TUPLE}(2, \delta, q)$ for any given η and δ if δ'' is sufficiently small. The second part of the proof makes use of Theorem 1.2 for 3-uniform hypergraphs. In Section 2 we sketch the proof of Theorem 1.1, explaining how we prove the implication $\mathcal{Q}'(\eta, \delta'', q) \Rightarrow \mathrm{TUPLE}(2, \delta, q)$. The proof of Theorem 1.2 is sketched in Section 3. We finish with some concluding remarks in Section 4.

2 Overview of the proof of Theorem 1.1

We start by defining some hypergraph properties. Let G be a 3-uniform hypergraph and let X, $Y \subset V(G)$. We say that (X, Y) satisfies property $\mathrm{DISC}(q, p, \varepsilon')$ in G if, for all $X' \subset \binom{X}{2}$ and $Y' \subset Y$, we have $\big||E_G(X', Y')| - q|X'||Y'|\big| \leq \varepsilon' p\binom{|X|}{2}|Y|$. Furthermore, if $(V(G), V(G))$ satisfies $\mathrm{DISC}(q, p, \varepsilon')$ in G, then we say that the hypergraph G satisfies $\mathrm{DISC}(q, p, \varepsilon')$. We say that (X, Y) satisfies property $\mathrm{PAIR}(q, p, \delta')$ in G if the following conditions hold:

$$\sum_{\{x_1, x_1'\} \in \binom{X}{2}} \big||N_G(\{x_1, x_1'\}, Y)| - q|Y|\big| \leq \delta' p\binom{|X|}{2}|Y|,$$

$$\sum_{\{x_1, x_1'\} \in \binom{X}{2}} \sum_{\{x_2, x_2'\} \in \binom{X}{2}} \big||N_G(\{x_1, x_1'\}, \{x_2, x_2'\}, Y)| - q^2|Y|\big| \leq \delta' p^2\binom{|X|}{2}^2|Y|,$$

where $N_G(\{x_1, x_1'\}, Y)$ denotes the set of vertices $y \in Y$ such that $\{x_1, x_1', y\} \in E(G)$ and $N_G(\{x_1, x_1'\}, \{x_2, x_2'\}, Y)$ denotes the set of vertices $y \in Y$ such that $\{x_1, x_1', y\} \in E(G)$ and $\{x_2, x_2', y\} \in E(G)$. Furthermore, if $(V(G), V(G))$ satisfies $\mathrm{PAIR}(q, p, \delta')$ in G, then we say that G satisfies $\mathrm{PAIR}(q, p, \delta')$.

Consider the setup of Theorem 1.1. The proof of Theorem 1.1 is divided into the following four parts. Below, for simplicity, we use $o(1)$ terms in our assertions, following standard practice in the area of quasi-randomness [2].

(i) $G_n \in \mathcal{Q}'(\eta, o(1), q)$ implies $(X, Y) \in \mathrm{DISC}(q, p, o(1))$ for large $X \subset \binom{V(G_n)}{2}$ and $Y \subset V(G_n)$;

(ii) $(X, Y) \in \mathrm{DISC}(q, p, o(1))$ implies $(X, Y) \in \mathrm{PAIR}(q, p, o(1))$;

(iii) $G_n \in \mathrm{PAIR}(q, p, o(1))$ implies $G_n \in \mathrm{TUPLE}(2, o(1), q)$;

(iv) Since $G_n \in \mathrm{BDD}(C, m, q)$ and $G_n \in \mathrm{TUPLE}(2, o(1), q)$, the counting result (Theorem 1.2) implies the conclusion of Theorem 1.1.

The jumbledness property of Γ is needed in the proof of items (i) and (ii). The proof of (i) is inspired by ideas in [10]. We partition large sets $X \subset \binom{V(G_n)}{2}$ and $Y \subset V(G_n)$ into sufficiently small pieces. Then we analyze the edge densities between these small pieces of X and Y. The proof of (ii) is quite long and is based on generalizations of results in [8]. The proof of (iii) is trivial and (iv) is just an application of Theorem 1.2.

3 Overview of the proof of Theorem 1.2

Consider the setup of Theorem 1.2. The next lemma allows us to replace property $\mathrm{TUPLE}(2, \delta, q)$ by $\mathrm{TUPLE}(d_H, \delta', q)$ in Theorem 1.2 as long as δ is sufficiently small.

Lemma 3.1 *For all $\delta' > 0$, $C > 1$ and integers $k, t \geq 2$, there exist $\delta, D > 0$ such that the following holds when $q = q(n) \geq Dn^{-1/t}$ and n is sufficiently large.*

If G_n is a k-uniform hypergraph such that $G_n \in \mathrm{BDD}(2, C, q)$, $G_n \in \mathrm{TUPLE}(2, \delta, q)$ and $|E(G_n)| = q\binom{n}{k}$, then $G_n \in \mathrm{TUPLE}(t, \delta', q)$.

Overview of the proof of Lemma 3.1. Fix $\delta' > 0$, $C > 1$ and integers $k, t \geq 2$. Consider $2 \leq r \leq t$ and let G_n and q be as in the statement of the theorem. We have to show that the conditions of a defect version of Cauchy–Schwarz inequality hold. In order to verify the validity of these conditions, we prove that if G_n satisfies $\mathrm{BDD}(C, 2, q)$, then G_n also satisfies a "version" of $\mathrm{BDD}(C, 2, q)$ for vertices instead sets of $k-1$ vertices. This is proved by induction on the size of the considered sets of vertices. We also have to prove that, for sufficiently small δ, property $\mathrm{TUPLE}(2, \delta, q)$ together with $\mathrm{BDD}(C, 2, q)$ implies a version of $\mathrm{TUPLE}(2, \delta, q)$ for vertices. This is proved by induction, Cauchy–Schwarz inequality and some counting arguments.

To sketch the proof of Theorem 1.2 we must consider the following definitions. Let $X \subset \binom{V(H)}{k-1}$. If f is an embedding of H into G_n, we denote by $f_{k-1}(X)$ the family of sets $\{f(x_1), \ldots, f(x_{k-1})\}$, for all $\{x_1, \ldots, x_{k-1}\} \in X$. Given $1 \leq r \leq k$ and a set $X = \{X_1, \ldots, X_r\}$, where $X_i = \{x_{i,1}, \ldots, x_{i,k-1}\} \in \binom{V(H)}{k-1}$ for $1 \leq i \leq r$, we define $X^{\mathrm{set}} = \{x_{1,1}, \ldots, x_{1,k-1}, \ldots, x_{r,1}, \ldots, x_{r,k-1}\}$.

Overview of the proof of Theorem 1.2. Fix k, m, ε and C. In our proof we need that $G_n \in \mathrm{TUPLE}(d_H, \delta', q)$ for a sufficiently small δ'. Let δ be given by an application of Lemma 3.1 with δ', C and $t = d_H$. Therefore, since $G_n \in \mathrm{TUPLE}(2, \delta, q)$, we conclude that $G_n \in \mathrm{TUPLE}(d_H, \delta', q)$.

Let H, G_n and q be as in the statement. Given $1 \leq h \leq m$, let $H_h = H[\{v_1, \ldots, v_h\}]$ where $\{v_1, \ldots, v_m\}$ is a d_H-degenerate ordering of $V(H)$. We will use induction on h to prove that $\big||\mathcal{E}(H_h, G_n)| - n^h q^{|E(H_h)|}\big| \leq \varepsilon n^h q^{|E(H_h)|}$.

First, by using that $G_n \in \mathrm{TUPLE}(d_H, \delta', q)$ and $G_n \in \mathrm{BDD}(C, D_H, q)$ we prove that most of the embeddings of H into G_n are induced and most of the embeddings $f \colon V(H_{h-1}) \to G_n$ are *clean*, where by "clean" we mean $\big|N_{G_n}(f_{k-1}(N_{H_h}(v_h))) - np^{d_{H_h}(v_h)}\big| < \delta' np^{d_{H_h}(v_h)}$ and $N^{\mathrm{set}}_{H_h}(v_h)$ is stable. Therefore, we can focus on clean and induced embeddings only.

Consider a clean and induced embedding f' from $V(H_{h-1})$ into G_n. Since H is linear and connector-free, $N^{\text{set}}_{H_h}(v_h)$ is stable in H_h. But since f' is induced, $f'(N^{\text{set}}_{H_h}(v_h))$ is stable in G_n. Since f' is clean, we also conclude that $|N_{G_n}(f'_{k-1}(N_{H_h}(v_h))) - nq^{d_{H_h}(v_h)}| < \delta' nq^{d_{H_h}(v_h)}$. To finish the proof, we count in how many ways we can extend f' to obtain an embedding of H_h into G_n.

4 Concluding remarks

Unfortunately, a version of Theorem 1.1 for k-uniform hypergraphs, for k larger than 3, present new difficulties and it will be considered elsewhere.

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