Extremal results for odd cycles in sparse pseudorandom graphs

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Abstract

We consider extremal problems for subgraphs of pseudorandom graphs. Our results implies that for (n, d, λ)-graphs Γ satisfying

$$\lambda^{2k-1} \ll \frac{d^{2k}}{n} (\log n)^{-2(k-1)(2k-1)}$$

any subgraph $G \subset \Gamma$ not containing a cycle of length $2k+1$ has relative density at most $\frac{1}{2} + o(1)$. Up to the polylog-factor the condition on λ is best possible and was conjectured by Krivelevich, Lee and Sudakov.

Keywords: odd cycles, extremal graph theory, pseudorandom graphs

1 Introduction and main result

For two graphs G and H, the *generalized Turán number*, denoted $\mathrm{ex}(G, H)$, is defined to be the largest number of edges an H-free subgraph of G may have. Here, a graph G is H-free if it contains no copy of H as a (not necessarily induced) subgraph. With this notation, the well known Erdős-Stone [8] theorem reads

$$\mathrm{ex}(K_n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2} \tag{1}$$

where $\chi(H)$ denotes the chromatic number of H.

The systematic study of extensions of the Erdős–Stone theorem arising from replacing K_n in (1) with a sparse random or a pseudorandom graph was initiated by Kohayakawa and collaborators (see, e.g., [9,10,11]). For random graphs such extensions were obtained recently in [7,15] (see also [4,14,6,13] for more recent developments).

Here, we continue the study for pseudorandom graphs. Roughly speaking, a pseudorandom graph is a graph whose edge distribution closely resembles that of a truly random graph of the same edge density. One way to formally capture this notion of pseudorandomness is through *eigenvalue separation*. A graph G on n vertices may be associated with a Boolean $n \times n$ adjacency matrix A. This matrix is symmetric and, hence, all its eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are real. If G is d-regular, then $\lambda_1 = d$ and $|\lambda_n| \leq d$ by the Perron-Frobenius theorem. The difference in order of magnitude between d and the *second eigenvalue* $\lambda(G) = \max\{\lambda_2, |\lambda_n|\}$ of G is often called the *spectral gap* of G. It is well known that the spectral gap provides a measure of control over the edge distribution of G. Roughly, the larger is the spectral gap the stronger is the resemblance between the edge distribution of G and that of the random graph $G(n, p)$, where $p = d/n$. This phenomenon led to the notion of (n, d, λ)-*graphs* by which we mean d-regular n-vertex graphs satisfying $\lambda(G) \leq \lambda$.

Turán type problems for sparse pseudorandom graphs were studied, e.g. in [11,16,5]. In this paper, we continue in studying extensions of the Erdős-Stone theorem for sparse host graphs and determine upper bounds for the

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generalized Turán number for odd cycles in sparse pseudorandom host graphs, i.e., $\mathrm{ex}(G, C_{2k+1})$ where G is a pseudorandom graphs and C_{2k+1} is the odd cycle of length $2k+1$.

Our work is related to work of Sudakov, Szabó, and Vu [16] who determined $\mathrm{ex}(G, K_t)$ for a pseudorandom graph G and $t \geq 3$. Their result may be viewed as the pseudorandom counterpart of Turán's theorem [19].

For any graph G, the trivial lower bound $\mathrm{ex}(G, C_{2k+1}) \geq e(G)/2$, where $e(G) = |E(G)|$, follows from the fact that every graph G contains a bipartite subgraph with at least half the edges of G. For $G \cong K_n$, this bound is tight and our result asserts that this bound remains essentially tight for sufficiently pseudorandom graphs.

Theorem 1.1 *Let $k \geq 1$ be an integer. If Γ is an (n, d, λ)-graph satisfying*

$$\lambda^{2k-1} \ll \frac{d^{2k}}{n} (\log n)^{-2(k-1)(2k-1)}, \tag{2}$$

then

$$\mathrm{ex}(\Gamma, C_{2k+1}) = \left(\frac{1}{2} + o(1)\right) \frac{dn}{2}.$$

For $k = 1$, the same problem was studied in [16]. In this case, we obtain the same result which is known to be best possible due to the construction of Alon [1]. For $k \geq 2$, Alon's construction can be extended as to fit for general odd cycles [2].This implies that for any $k \geq 2$, up the polylog-factor, the condition (2) is best possible and confirms a conjecture of Krivelevich, Lee and Sudakov [12]. Theorem 1.1 is a consequence of Theorem 1.2 stated below for the so called *jumbled* graphs. We recall this notion of pseudorandomness which can be traced back to Thomason [18].

Given $p = p(n)$ and $\gamma = \gamma(n)$, we say that an n-vertex graph Γ is (p, γ)-*jumbled* if for all disjoint $X, Y \subset V(\Gamma)$ we have

$$|e(X, Y) - p|X||Y|| \leq \gamma\sqrt{|X||Y|}\,.$$

The following is our main result.

Theorem 1.2 *For every integer $k \geq 1$ and every $\delta > 0$ there exists a $\gamma > 0$ such that for every sequence of densities $p = p(n)$ there exists an n_0 such that for any $n \geq n_0$ the following holds.*

If Γ is an n-vertex (p, β)-jumbled graph satisfying

$$\beta \leq \gamma p^{1+\frac{1}{(2k-1)}} n \log^{-2(k-1)} n, \tag{3}$$

then

$$\mathrm{ex}(\Gamma, C_{2k+1}) < \left(\frac{1}{2} + \delta\right) p \binom{n}{2}.$$

By the so called *expander mixing lemma* [3,17] an (n, d, λ)-graph is (p, β)-jumbled with $p = d/n$ and $\beta = \lambda$. Hence, Theorem 1.2 indeed implies Theorem 1.1.

2 Sketch of the proof of Theorem 1.2

Theorem 1.2 easily follows from Lemmas 2.1 and 2.2 stated below. To state Lemma 2.1, we employ the following notation.

For a graph G and disjoint vertex sets $X, Y \subseteq V(G)$, we write $G[X, Y]$ to denote the bipartite subgraph of G induced by the bipartition $X \cup Y$. For a graph R and a positive integer m, we write $R(m)$ to denote the graph obtained by replacing every vertex $i \in V(R)$ with a set of vertices V_i of size m and adding the complete bipartite graph between V_i and V_j whenever $ij \in E(R)$. A spanning subgraph of $R(m)$ is called an *$R(m)$-graph*. In addition, such a graph, say $G \subseteq R(m)$, is called *(α, p, ε)-degree-regular* if $\deg_{G[V_i,V_j]}(v) = (\alpha \pm \varepsilon)pm$ holds whenever $ij \in E(R)$ and $v \in V_i \cup V_j$.

The following lemma essentially asserts that under a certain assumption of jumbledness, a relatively dense subgraph of a sufficiently large (p, β)-jumbled graph contains a degree-regular C_ℓ(m)-graph with large m.

Lemma 2.1 *For any integer $\ell \geq 3$, all $\varrho > 0$, $\alpha_0 > 0$ and $0 < \varepsilon < \alpha_0$ there exist a $\nu > 0$ and a $\gamma > 0$ such that for every sequence of densities $p = p(n) \gg \log n/n$ there exists an n_0 such that for every $n \geq n_0$ the following holds.*

Let Γ be an n-vertex (p, β)-jumbled graph with $\beta = \beta(n) \leq \gamma p^{1+\varrho} n$ and let $G \subset \Gamma$ be a subgraph of Γ satisfying $e(G) \geq \alpha_0 p\binom{n}{2}$. Then, there exists an $\alpha \geq \alpha_0$ such that G contains an (α, p, ε)-degree-regular $C_\ell(\nu n)$-graph as a subgraph. □

Equipped with Lemma 2.1, we focus on large degree-regular $C_\ell(m)$-graphs hosted in a sufficiently jumbled graph Γ. In this setting, we shall concentrate on odd cycles in Γ that have all but one of their edges in the hosted $C_\ell(m)$-graph. The remaining edge belongs to Γ. The first part of Lemma 2.2 stated below provides a lower bound for the number of such configurations (see (5)). We now make this precise.

Fix a vertex labeling of C_{2k+1}, say, $(u_k, \ldots, u_1, w, v_1, \ldots, v_k)$. For a jumbled graph Γ (as in Lemma 2.2), let $H \subseteq \Gamma$ be a $C_{2k+1}(m)$-graph with the corresponding vertex partition $(U_k, \ldots, U_1, W, V_1, \ldots, V_k)$. By $\mathcal{C}(H, \Gamma)$ we denote the set of all cycles of length $(2k+1)$ of the form $(u'_k, \ldots, u'_1, w', v'_1, \ldots, v'_k)$ such that $w' \in W, v'_i \in V_i$, $u'_i \in U_i$, $v'_k u'_k \in E(\Gamma)$, and all edges other than $v'_k u'_k$ in $E(H)$. In other words, a member of $\mathcal{C}(H, \Gamma)$ is a cycle of Γ of length $2k+1$ with the additional requirement that the labeled edge $v'_k u'_k$ connects the ends of the path of length $2k$ in H. If $v'_k u'_k$ is contained in H, then clearly, H contains a C_{2k+1}.

For a real number $\mu > 0$, an edge of $\Gamma[V_k, U_k]$ is called *μ-saturated* if such is contained in at least $p(\mu p m)^{2k-1}$ members of $\mathcal{C}(H, \Gamma)$. A cycle in $\mathcal{C}(H, \Gamma)$ containing a μ-saturated edge is called a *μ-saturated cycle*. We write $\mathcal{S}(\mu, H, \Gamma)$ to denote the set of μ-saturated cycles in $\mathcal{C}(H, \Gamma)$. To motivate the definition of μ-saturated edges, note that we expect that an edge of $\Gamma[U_k, V_k]$ extends to $(\alpha p)^{2k} m^{2k-1}$ members of $\mathcal{C}(H, \Gamma)$. For $\mu \approx \alpha$, a μ-saturated edge overshoots this expectation by a factor of $1/\alpha$.

Lemma 2.2 *For any integer $k \geq 1$ and all reals $0 < \nu, \alpha_0 \leq 1$, and $0 < \varepsilon \leq \alpha_0/3$ there exists a $\gamma > 0$ such that for every sequence of densities $p = p(n)$ there exists an n_0 such that for any $n \geq n_0$ the following holds.*

If Γ is (p, β)-jumbled n-vertex graphs with

$$\beta = \beta(n) \leq \gamma p^{1+\frac{1}{2k-1}} n \log^{-2(k-1)} n, \tag{4}$$

then for any $m \geq \nu n$ and any $\alpha \geq \alpha_0$ an (α, p, ε)-degree-regular $C_{2k+1}(m)$-graph $H \subseteq \Gamma$ satisfies

$$|\mathcal{C}(H, \Gamma)| \geq (\alpha - 2\varepsilon)^{2k} (pm)^{2k+1} \qquad \text{and} \tag{5}$$

$$|\mathcal{S}(\alpha + 2\varepsilon, H, \Gamma)| \leq (3\varepsilon)^{2k} (pm)^{2k+1}. \tag{6}$$

□

With Lemma 2.1 and Lemma 2.2 at hand Theorem 1.2 easily follows. Let G and Γ be as in Theorem 1.2. Using Lemma 2.1 we find an (α, p, ε)-degree-regular $C_\ell(\nu n)$-graph with vertex partition $(U_k, \ldots, U_1, W, V_1, \ldots, V_k)$ as a subgraph of G where $\alpha \geq 1/2$. By (5) we find at least $(\alpha - 2\varepsilon)^{2k} (pm)^{2k+1}$ cycles of the form $(u'_k, \ldots, u'_1, w', v'_1, \ldots, v'_k)$ such that $w' \in W, v'_i \in V_i$, $u'_i \in U_i$, $v'_k u'_k \in E(\Gamma)$ where all but the edge $v'_k u'_k$ of the cycle is in H. Call such an edge a *forbidden edge* and we wish to show that the set $F \subset \Gamma[V_k, U_k]$ of forbidden edges intersects with $E(H)$ which would prove the existence of a cycle of length $2k+1$ in $H \subset G$. Choosing ε sufficiently small depending on δ we obtain

$$|F| \geq \frac{|\mathcal{C}(H, \Gamma) \setminus \mathcal{S}(\alpha + 2\varepsilon, H, \Gamma)|}{p\,(\alpha + 2\varepsilon)^{2k-1} (pm)^{2k-1}} \geq \frac{(\alpha - 5\varepsilon)^{2k}}{(\alpha + 2\varepsilon)^{2k-1}} pm^2 > \left(\alpha - \frac{\delta}{2}\right) pm.$$

Hence, with $\alpha \geq 1/2$, we derive

$$|F| + e(H[V_k, U_k]) \geq (2\alpha + \delta/2)pm^2 \geq (1 + \delta/2)pm^2 > e(\Gamma[V_k, U_k])$$

and F must intersect $E(H)$.

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