Note on forcing pairs

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Abstract

The notion of forcing pairs is located in the study of quasi-random graphs. Roughly speaking, a pair of graphs (F, F') is called forcing if the following holds: Suppose for a sequence of graphs (G_n) there is a $p > 0$ such that the number of copies of F and the number of copies of F' in every graph G_n of the sequence (G_n) is approximately the same as the expected value in the random graph $G(n, p)$, then the sequence of graphs (G_n) is quasi-random in the sense of Chung, Graham and Wilson. We describe a construction which, given any graph F with at least one edge, yields a graph F' such that (F, F') forms a forcing pair.

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1 Introduction

We study forcing pairs, a notion closely related to quasi-random graphs. The systematic study of quasi-random graphs was initiated by Thomason [8,9] and its main goal is to provide "deterministic" properties which capture the characteristics of random graphs. One such property is given by the notion of uniform edge distribution which we also refer to as low discrepancy. We say a given sequence of graphs (G_n) with $|V(G)| = n$ has low discrepancy (denoted by DISC_p), if

$$e(U) = p\binom{|U|}{2} + o(n^2) \quad \text{for all } U \subset V(G). \tag{1}$$

In [1] Chung, Graham and Wilson (building on the work of others) gave a list of so-called quasi-random properties which are equivalent to DISC_p, meaning every sequence (G_n) that satisfies any of these properties must satisfy all of them. In particular, it is shown that the following property MIN_p is equivalent to DISC_p. Let (G_n) be a sequence of graphs and let $N_F(G_n)$ denote the number of labeled copies of F in G_n, we say (G_n) satisfies MIN_p if

$$N_{K_2}(G_n) \geq pn^2 - o(n^2) \quad \text{and} \quad N_{C_4}(G_n) \leq p^4 n^4 + o(n^4). \tag{2}$$

For constant $p > 0$ the property MIN_p is almost surely satisfied by the random graph $G(n,p)$ and the equivalence between property (1) and (2) particularly implies that a lower bound on K_2 (i.e., on the edge density of every graph G_n of the sequence) and a corresponding upper bound on the number of C_4 force (G_n) to have uniform edge distribution.

This motivates the question which other pairs of graphs (replacing (K_2, C_4)) have this property and gives rise to the notion of forcing pairs.

Definition 1.1 A pair of graphs (F, F') is called **forcing** if every sequence of graphs (G_n) with $|V(G_n)| = n$ which satisfies

$$N_F(G_n) \geq p^{e(F)} n^{v(F)} - o(n^{v(F)}) \quad \text{and} \quad N_{F'}(G_n) \leq p^{e(F')} n^{v(F')} + o(n^{v(F')}) \tag{3}$$

also satisfies DISC_p.

It is known that a graph sequence, which satisfies DISC_p also satisfies the condition (3) of Definition 1.1. Hence, generalizing the case (K_2, C_4), any forcing pair (F, F') gives rise to a quasi-random property in the sense of Chung, Graham, and Wilson due to (3) of Definition 1.1.

In fact, it is well known that every sequence (G_n) which satisfies DISC_p, also satisfies $N_F^*(G_n) = (1 \pm o(1)) p^{e(F)} (1-p)^{\binom{v(F)}{2} - e(F)} n^{v(F)}$ for every fixed graph F, where $N^*_F(G_n)$ denotes the number of labeled, induced copies of F in G_n. Due to this the notion of forcing pairs (or families of graphs) varies in the literature. For example, in the original work of Chung, Graham, and Wilson in [1] the condition in (3) is replaced by appropriate bounds on N^*_F and $N^*_{F'}$ and in [2,5] matching upper and lower bounds on $N_F(G_n)$ are required.

Forcing pairs other than (K_2, C_4) were discovered in [1,2,3,4,7]. In particular, in [3] the forcing pairs involving non-bipartite graphs were found. In this note we prove that for every graph F with at least one edge there is a graph F' such that (F, F') forms a forcing pair. The proof is constructive and indeed, for a given graph F the graph F' is given by the following construction from [3].

For a k-partite graph A with vertex classes $X_1, \dots, X_k$ and $i \in [k]$ we define the *doubling* $\mathrm{db}_i(A)$ *of* A *around vertex class* X_i to be the graph obtained from A by taking two disjoint copies of A and identifying the vertices of X_i. More formally, $\mathrm{db}_i(A)$ is the k-partite graph with vertex classes $Y_1, \dots, Y_k$, where $Y_i = X_i$ and for $j \neq i$ we have $Y_j = X_j \dot\cup \tilde{X}_j$ with $\tilde{X}_j = \{\tilde{x} \mid x \in X_j\}$. Thus $\tilde{x}$ denotes the copy of the vertex x. Moreover, the edge set of $\mathrm{db}_i(A)$ is given by

$$E(\mathrm{db}_i(A)) = E(A) \dot\cup \{\tilde{x}_j \tilde{x}_{j'} \colon\ x_j x_{j'} \in E(A)\} \dot\cup \{x_i \tilde{x}_j \colon\ x_i x_j \in E(A)\}.$$

We start with the graph F and consider it as a $v(F)$-partite graph with every vertex lying in its own partition class. Then a graph F' which makes (F, F') a forcing pair is obtained by successively doubling the $v(F)$-partite graph F around the classes $i = 1, 2, \dots, k$, i.e.

$$M(F) = \mathrm{db}_k(\mathrm{db}_{k-1}(\dots \mathrm{db}_1(F) \dots)).$$

It can be shown that the order of the doubling operations has no effect on $M(F)$, i.e., $M(F)$ is independent from the initial labeling of the vertices of F.

Theorem 1.2 *For every graph F with $e(F) \geq 1$ is $(F, M(F))$ a forcing pair.*

2 Auxiliary results and proof of the main theorem

We introduce and sketch a proof of the main auxiliary lemma (Lemma 2.1). In Section 2.2 we deduce Theorem 1.2 from Lemma 2.1 and a result of Simonovits and Sós from [6].

2.1 Main auxiliary lemma

Instead of dealing with labeled copies (i.e., injective homomorphisms) of a graph A in G we will consider all graph homomorphism ($\mathrm{Hom}(A, G)$) from A into G. Let A be a k-partite graph with the partition classes $X_1, \dots, X_k$ and G be a k-partite graph with the partition classes $V_1, \dots, V_k$. For subsets $U_i \subseteq V_i$, $i \in [k]$, we denote by $\mathrm{Hom}(A, G, U_1, \dots, U_k)$ those homomorphisms of A into G that map vertices from X_i to U_i. The main auxiliary result is given by the following.

Lemma 2.1 *For every graph F with $k = v(F)$, every $p > 0$, and every $\varepsilon > 0$ there exist a $\delta > 0$ and n_0 such that for every k-partite graph $G = (V, E)$ with vertex classes $V_1, \dots, V_k$, each of size $n \geq n_0$, which satisfies*

$$|\mathrm{Hom}(F, G, V_1, V_2, \dots, V_k)| \geq p^{e(F)} n^k - \delta n^k$$

and

$$|\mathrm{Hom}(M(F), G, V_1, V_2, \dots, V_k)| \leq p^{2^k \cdot e(F)} n^{k 2^{k-1}} + \delta n^{k 2^{k-1}},$$

we have that for all families of subsets $U_i \subseteq V_i$ with $i \in [k]$

$$|\mathrm{Hom}(M(F), G, U_1, \dots, U_k)| = p^{2^k \cdot e(F)} \prod_{i=1}^{k} |U_i|^{2^{k-1}} \pm \varepsilon n^{k 2^{k-1}}.$$

Proof of Lemma 2.1 (Sketch). In a first step we show that

$$|\mathrm{Hom}(M(F), G, V_1, \dots, V_{k-1}, U_k)| \leq p^{2^k e(F)} n^{(k-1) 2^{k-1}} |U_k|^{2^{k-1}} + \delta' n^{k 2^{k-1}} \quad (4)$$

for some $\delta' \to 0$ as $\delta \to 0$.

Let $M_i = \mathrm{db}_i(\mathrm{db}_{i-1}(\dots(\mathrm{db}_1(F))\dots))$ and let G be a k-partite graph as stated in the lemma. Owing to the lower bound on $|\mathrm{Hom}(F, G, V_1, \dots, V_k)|$, it follows from the Cauchy-Schwarz inequality that $|\mathrm{Hom}(M_{k-1}, G, V_1, \dots, V_k)|$ is at least what we would expect if the edges were chosen independently with probability p. Here, we crucially use the fact that M_{k-1} arises from doubling operations.

Subsequently, we apply the following well-known fact. If $a_1, \dots, a_N$ satisfy

$$\sum_{i=1}^{N} a_i \geq (1 - o(1)) a N \quad \text{and} \quad \sum_{i=1}^{N} a_i^2 \leq (1 + o(1)) a^2 N,$$

then almost all a_i are roughly a. Let X_k be the k-th vertex partition class of M_{k-1} and let N be the number of $|X_k|$-tuples in V_k. For $i = 1, \dots, N$ we define a_i to be the number of homomorphisms $\varphi \in \mathrm{Hom}(M_{k-1}, G, V_1, \dots, V_k)$ which maps X_k to the i-th $|X_k|$-tuple in V_k. Hence, $\sum_{i=1}^{N} a_i$ corresponds to $|\mathrm{Hom}(M_{k-1}, G, V_1, \dots, V_k)|$ which is bounded from below due to the assumption on G. Moreover, the upper bound on $|\mathrm{Hom}(M(F), G, V_1, \dots, V_k)|$ given by the assumption on G translates to a corresponding bound on $\sum_{i=1}^{N} a_i^2$. Hence, we obtain that almost all a_i is roughly the average a, which is what we would expect if the edges were chosen independently randomly with probability p. In particular, almost all $|X_k|$-tuples from V_k and consequently almost all $|X_k|$-tuples from U_k satisfy this property from which (4) can be derived using the fact that $M(F) = M_k$ arises by doubling of M_{k-1} around the vertex class X_k.

Next we repeat the same reasoning above iteratively to obtain similar upper bounds for

$$|\mathrm{Hom}(M(F), G, V_1, \dots, V_{i-1}, U_i, \dots, U_k)| \quad \text{for } i = k-1, k-2, \dots, 1.$$

For $i = 1$ this then yields the conclusion of Lemma 2.1. To this end note that we need an appropriate lower bound for $|\mathrm{Hom}(F, G, V_1, \dots, V_i, U_{i+1}, \dots, U_k)|$, which is not provided as a direct assumption of the lemma. However, this can be obtained by a similar argument as above, using the fact that the graph $M(F)$ is independent of the ordering of the $v(F)$ vertex classes. We omit the details here. $\square$

2.2 Proof of the Theorem 1.2

Theorem 1.2 will follow from Lemma 2.1 and the following result of Simonovits and Sós [6].

Theorem 2.2 *For every $p > 0$, every graph F with $e(F) \geq 1$, and every $\varepsilon > 0$ there exist $\delta > 0$ and n_0 such that the following is true. If $G = (V, E)$ is a graph with $|V| = n \geq n_0$ vertices such that $N_F(U) = p^{e(F)}|U|^{v(F)} \pm \delta n^{v(F)}$ for every subset $U \subseteq V$ then $e(U) = p\binom{|U|}{2} \pm \varepsilon n^2$ for every subset $U \subseteq V$.* $\square$

Proof of Theorem 1.2. Note that it is sufficient to prove the following. For a given $\varepsilon > 0$ there exists a $\delta > 0$ such that any n-vertex graph G on n vertices satisfying (3) with the error term $o(n^{v(F)})$ replaced by $\delta n^{v(F)}$, it is true that G satisfies $e(U) = p\binom{|U|}{2} \pm \varepsilon n^2$ for all $U \subset V$ (which is DISC_p with the error term $o(n^2)$ replaced by δn^2). Further, note that we can safely replace $N_F(G)$ and $N_{M(F)}(G)$ by $|\mathrm{Hom}(F, G)|$ and $|\mathrm{Hom}(M(F), G)|$, as the number of non-injective homomorphisms is of smaller order and hence negligible for sufficiently large n.

For a given graph G, we define the k-partite graph $G^{(k)}$ by taking k copies of $V(G)$ and connecting vertices between different copies if they form an edge in the original graph G. Formally, let $V(G^{(k)}) = [k] \times V(G)$ and $\{(i,v),(j,u)\}$ is an edge in $G^{(k)}$ whenever $\{u,v\} \in E(G)$ and $i \neq j$.

We apply Lemma 2.1 to the graph $G^{(k)}$ and the pair $(F, M(F))$ with sufficiently small error term. From the conclusion of the Lemma 2.1 we obtain that G satisfies the presumption of Theorem 2.2. This holds since every homomorphism from $\mathrm{Hom}(M(F), G[U])$ corresponds to a homomorphism from $\mathrm{Hom}(M(F), G^{(k)}, (\{1\} \times U, \dots, \{k\} \times U))$. Applying Theorem 2.2 we obtain $e(U) = p\binom{|U|}{2} \pm \varepsilon n^2$ for all $U \subset V$ and Theorem 1.2 follows. $\square$

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