Every Monotone 3-Graph Property is Testable

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Let $k \geq 2$ be a fixed integer and $\mathcal{P}$ be a **property** of k-uniform hypergraphs. In other words, $\mathcal{P}$ is a (typically infinite) family of k-uniform hypergraphs and we say a given hypergraph H **satisfies** $\mathcal{P}$ if $H \in \mathcal{P}$. For a given constant $\eta > 0$ a k-uniform hypergraph H on n vertices is η**-far** from $\mathcal{P}$ if no hypergraph obtained from H by changing (adding or deleting) at most ηn^k edges in H satisfies $\mathcal{P}$. More precisely, H is η-far from $\mathcal{P}$ if no hypergraph G with $|E(G) \triangle E(H)| \leq \eta n^k$ satisfies $\mathcal{P}$. This is a natural measure of how far the given hypergraph H is to satisfy the property $\mathcal{P}$.

We consider randomized algorithms which for an input hypergraph H on the vertex set $V(H) = \{1, \ldots, n\}$ are able to make queries whether a given k-tuple of vertices spans an edge in H. For a given decidable property $\mathcal{P}$ and a constant $\eta > 0$, such an algorithm will be called a **tester for** $\mathcal{P}$ if it can distinguish with, say probability $\frac{2}{3}$, whether H satisfies $\mathcal{P}$ or is η-far from it. If a property $\mathcal{P}$ has a tester whose number of queries is bounded by a function of η and $\mathcal{P}$ but is independent on the number of vertices of the input hypergraph H, the property is called **testable**. Perhaps surprisingly, many natural properties can be proved to be testable.

The general notion of property testing was introduced by R. Rubinfeld and M. Sudan in [13]. In [8], O. Goldreich, S. Goldwasser and D. Ron initiated

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the study of property testing for combinatorial structures. They showed that several graph properties, including t-colorability, are testable.

Some ideas of property testing were already implicitly present long before the notion of a tester was actually defined. Thus several results of property testing take their roots in classical graph or hypergraph theorems. For example, the testability of triangle freeness is a consequence of a well know result of I. Z. Ruzsa and E. Szemerédi [14]. The result from [10] implies that t-colorability is testable (see also [1]). The property of not containing a fixed graph F as a non induced subgraph also falls into this category. It was indirectly proved to have a tester by P. Erdős, P. Frankl and V. Rödl [5]. The proof from [14], as well as, its generalization in [5] are based on Szemrédi's regularity lemma [15].

Note that the property of being t-colorable as well as the property of not containing a fixed graph F as a (non induced) subgraph are monotone. Here we call a property $\mathcal{P}$ **monotone**, if it is closed by taking (not necessarily induced) sub-hypergraphs. In other words if H satisfies $\mathcal{P}$, then any sub-hypergraph of H satisfies $\mathcal{P}$ as well. In this case, for H to be η-far from $\mathcal{P}$, it is enough to verify that no sub-hypergraph G of H with $|E(H) \setminus E(G)| < \eta n^k$ satisfies $\mathcal{P}$.

Another example of a testable, but not necessarily monotone property, is the family of all graphs not containing a fixed graph F as an induced subgraph. The testability of those properties for a given graph F was proved by N. Alon et al. [2]. This is harder to prove than the non induced version and the proof relies on a strengthened version of Szeméredi regularity lemma. For more results on property testing see e.g. [6,7,12].

Although many of the combinatorial results on testing were proved for graphs, some problems have been studied for k-uniform hypergraphs. For example in [4], the authors prove that t-colorability for uniform hypergraph is a testable (monotone) property. Y. Kohayakawa, B. Nagle and V. Rödl considered in [9] an analogous result to [2] for 3-uniform hypergraphs.

Recently, N. Alon and A. Shapira [3] proved that every monotone graph property is testable. The tester built in their proof has the further property that, for any $H \in \mathcal{P}$, it confirms with probability 1 that H satisfies $\mathcal{P}$. In other words, whenever the input graph H satisfies $\mathcal{P}$, the algorithm will be correct with probability 1. This type of tester is said to have **one-sided error**. Altogether the result in [3] means that for any monotone property $\mathcal{P}$ there exists a tester which after at most $f(\eta, \mathcal{P})$ queries, comes to the following conclusion:

(i) If $H \in \mathcal{P}$, then the tester confirms it with probability 1.

(ii) If H is η-far from $\mathcal{P}$, then the tester outputs that $H \notin \mathcal{P}$ with probability $\frac{2}{3}$.

The proof of N. Alon and A. Shapira is also based on the strengthened version of the regularity lemma used in [2].

In [3], the authors raised the question whether it is possible to extend their result to k-uniform hypergraphs. The aim of this note is to announce a positive answer for 3-uniform hypergraphs.

Theorem 0.1 *Any monotone property of* 3*-uniform hypergraphs is testable with one-sided error.*

Monotone properties can be described by a family of forbidden hypergraphs, i.e., for a monotone property $\mathcal{P}$ there exists a family of hypergraphs $\mathcal{F} = \mathcal{F}_{\mathcal{P}}$ such that $\mathcal{P} = \mathrm{Forb}(\mathcal{F})$, where $\mathrm{Forb}(\mathcal{F})$ is the family of those hypergraphs not containing any element of $\mathcal{F}$ as a sub-hypergraph. Theorem 0.1 is an immediate consequence of the following result.

Theorem 0.2 *Let $\mathcal{F}$ be a family of 3-uniform hypergraphs and $\mathcal{P} = \mathrm{Forb}(\mathcal{F})$. For all $\eta > 0$ there are constants C and $n_0 \in \mathbb{N}$ such that the following holds:*

If H is a 3-uniform hypergraph on $n \geq n_0$ vertices which is η-far from satisfying $\mathcal{P}$, then there exists a hypergraph $F_0 \in \mathcal{F}$ with $|V(F_0)| < C$ such that the number of copies of F_0 in H is $\Omega(n^{|V(F_0)|})$.

The main philosophy of our proof of Theorem 0.2 is similar to [3]. However, instead of Szeremédi's regularity lemma, we use a hypergraph regularity lemma from [11]. Using this lemma in the case $k = 3$, we develop an analogue for 3-uniform hypergraphs to the strengthened variant of Szeméredi regularity lemma proved in [2]. Currently our proof is fairly involved and we believe that to extend it to general k is more a technical than a principle issue. This is partly supported by the fact that the crucial hypergraph regularity was proved for general k-uniform hypergraphs.

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