# GENERALIZATIONS OF THE REMOVAL LEMMA*

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Abstract. Ruzsa and Szemerédi established the *triangle removal lemma* by proving that: *For every $\eta > 0$ there exists $c > 0$ so that every sufficiently large graph on $n$ vertices, which contains at most $cn^3$ triangles can be made triangle free by* removal *of at most $\eta\binom{n}{2}$ edges.* More general statements of that type regarding graphs were successively proved by several authors. In particular, Alon and Shapira obtained a generalization (which extends all the previous results of this type), where the triangle is replaced by a possibly infinite family of graphs and containment is induced.

In this paper we prove the corresponding result for $k$-uniform hypergraphs and show that: *For every family $\mathscr{F}$ of $k$-uniform hypergraphs and every $\eta > 0$ there exist constants $c > 0$ and $C > 0$ such that every sufficiently large $k$-uniform hypergraph on $n$ vertices, which contains at most $cn^{v_F}$ induced copies of any hypergraph $F \in \mathscr{F}$ on $v_F \leq C$ vertices can be changed by adding and deleting at most $\eta\binom{n}{k}$ edges in such a way that it contains no induced copy of any member of $\mathscr{F}$.*

## 1. Introduction

Answering a question of Brown, T. Sós, and Erdős [7, 30] Ruzsa and Szemerédi [27] established the *triangle removal lemma*. They proved that every graph which does not contain many triangles can be "made easily" triangle free.

**Theorem 1** (Triangle removal lemma)**.** *For every $\eta > 0$ there exists $c > 0$ and $n_0$ so that every graph $G$ on $n \geq n_0$ vertices, which contains at most $cn^3$ triangles can be made triangle free by removing at most $\eta\binom{n}{2}$ edges.* □

More general statements of that type regarding graphs were successively proved by several authors in [1, 2, 3, 10]. In particular, the result of Alon and Shapira in [2] is a generalization, which extends all the previous results of this type, where the triangle is replaced by a possibly infinite family of graphs and containment is induced. The main result of the present paper is Theorem 6, which is an extension of the result of Alon and Shapira from graphs to $k$-uniform hypergraphs.

Before we state Theorem 6 we discuss some of the known extensions of the Ruzsa–Szemerédi theorem for graphs and hypergraphs in more detail.

### 1.1. Previous work.

A $k$-uniform hypergraph $H^{(k)}$ on the vertex set $V$ is some family of $k$-element subsets of $V$, i.e., $H^{(k)} \subseteq \binom{V}{k}$. Note that we identify hypergraphs with their edge set and we write $V(H^{(k)})$ for the vertex set. In this paper we only consider uniform hypergraphs, where the uniformity is some fixed number independent of the size of the hypergraph. We usually indicate the uniformity by a superscript.

A possible generalization of Theorem 1 to hypergraphs was suggested in [10, Problem 6.1]. The first result in this direction was obtained by Frankl and Rödl [11] who extended Theorem 1 to 3-uniform hypergraphs with the triangle replaced by

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$K_4^{(3)}$ – the complete 3-uniform hypergraph on 4 vertices. The general result, which settles the conjecture from [10] was recently obtained independently by Gowers [15] and Nagle, Skokan and authors [20, 25, 26] and subsequently by Tao in [33].

**Theorem 2** (Removal lemma). *For all $k$-uniform hypergraphs $F^{(k)}$ on $\ell$ vertices and and every $\eta > 0$ there exist $c > 0$ and $n_0$ so that the following holds.*

*Suppose $H^{(k)}$ is a $k$-uniform hypergraph on $n \geq n_0$ vertices. If $H^{(k)}$ contains at most $cn^\ell$ copies of $F^{(k)}$, then one can delete $\eta\binom{n}{k}$ edges from $H^{(k)}$ so that the resulting sub-hypergraph contains no copy of $F^{(k)}$.* □

Theorem 2 implies Szemerédi's theorem [31] as well as its multidimensional extensions due to Furstenberg and Katznelson [12, 13] (see, e.g., [15, 21, 24, 26, 28, 29, 33] for details).

Similarly, as all known proofs of Theorem 1 are based on *Szemerédi's regularity lemma* [32] (see, e.g., [17]), all proofs of Theorem 2 rely on hypergraph generalizations of the regularity lemma (see, e.g., [15, 20, 22, 23, 25, 33]).

One possible generalization of Theorem 2 is to replace the single hypergraph $F^{(k)}$ by a possibly infinite family $\mathscr{F}$ of $k$-uniform hypergraphs. Such a result was first proved for graphs by Alon and Shapira [3] in the context of property testing. For a family of graphs $\mathscr{F}$ consider the class $\mathrm{Forb}(\mathscr{F})$ of all graphs $H$ containing no member of $\mathscr{F}$ as a subgraph. Clearly $\mathrm{Forb}(\mathscr{F})$ is monotone, i.e., if $H \in \mathrm{Forb}(\mathscr{F})$ and $H'$ is a subgraph of $H$ (obtained from $H$ by successive vertex and edge deletions), then $H' \in \mathrm{Forb}(\mathscr{F})$. Moreover, it is easy to see that for every monotone family of graphs $\mathscr{P}$ (so-called monotone property $\mathscr{P}$) there exists a family $\mathscr{F}$ such that $\mathscr{P} = \mathrm{Forb}(\mathscr{F})$. Alon and Shapira proved the following in [3].

**Theorem 3.** *For every (possibly infinite) family of graphs $\mathscr{F}$ of graphs and every $\eta > 0$ there exist constants $c > 0$, $C > 0$, and $n_0$ such that the following holds.*

*Suppose $H$ is a graph on $n \geq n_0$ vertices. If for every $\ell = 1, \dots, C$ and every $F \in \mathscr{F}$ on $\ell$ vertices, $H$ contains at most $cn^\ell$ copies of $F$, then one can delete $\eta\binom{n}{2}$ edges from $H$ so that the resulting subgraph $H'$ contains no copy of any member of $\mathscr{F}$, i.e., $H' \in \mathrm{Forb}(\mathscr{F})$.* □

Clearly, Theorem 2 for $k = 2$ is equivalent to Theorem 3 in the special case when $\mathscr{F}$ consists of only one graph. While for finite families $\mathscr{F}$ Theorem 3 can be proved along the lines of the proof of Theorem 2 (or be deduced from Theorem 2 directly), for infinite families $\mathscr{F}$ the proof of Theorem 3 is more sophisticated.

Perhaps one of the earliest results of this nature was obtained by Bollobás, Erdős, Simonovits, and Szemerédi [5], who essentially proved Theorem 3 for the special family $\mathscr{F}$ of blow-up's of odd cycles. In [8] answering a question of Erdős (see, e.g., [9]) Duke and Rödl generalized the result from [5] and proved Theorem 3 for the families of $(r+1)$-chromatic graphs $r \geq 2$.

The proof of Theorem 3 for arbitrary families $\mathscr{F}$ relies on a strengthened version of Szemerédi's regularity lemma, which was obtained by Alon, Fischer, Krivelevich, and M. Szegedy [1] by iterating the regularity lemma for graphs.

Recently, Theorem 3 was extended by Avart and authors in [4] from graphs to hypergraphs. The proof in [4] follows the approach of Alon and Shapira and is based on two successive applications of the hypergraph regularity lemma from [22].

Another natural variant of Theorem 2 would be an *induced* version. For graphs this was first considered by Alon, Fischer, Krivelevich, and M. Szegedy [1]. Note that in this case in order to obtain an induced $F$-free graph, we may need to not only remove, but also to add edges.

**Theorem 4.** *For all graphs $F$ on $\ell$ vertices and and every $\eta > 0$ there exist $c > 0$ and $n_0$ so that the following holds.*

*Suppose $H$ is a graph on $n \geq n_0$ vertices. If $H$ contains at most $cn^\ell$ induced copies of $F$, then one can change $\eta\binom{n}{2}$ pairs from $V(H)$ (deleting or adding the edge) so that the resulting graph $H'$ contains no induced copy of $F$.* $\square$

An extension of Theorem 4 to 3-uniform hypergraphs was obtained by Kohayakawa, Nagle, and Rödl in [16].

In [2] Alon and Shapira proved a common generalization of Theorem 3 and Theorem 4, extending Theorem 4 from one forbidden induced graph $F$ to a forbidden family of induced graphs $\mathscr{F}$. The aim of this paper is to extend their result to $k$-uniform hypergraphs.

1.2. **Main result.** For a family of $k$-uniform hypergraphs $\mathscr{F}$, let $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ be the family of all hypergraphs $H^{(k)}$ which contain no induced copy of any member of $\mathscr{F}$. Clearly, $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ is a *hereditary* family (or *hereditary property*) of hypergraphs, i.e., if $H^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ and $\tilde{H}^{(k)}$ is an induced sub-hypergraph of $H^{(k)}$, then $\tilde{H}^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$.

**Definition 5** (**$\eta$-far**). *For a constant $\eta \geq 0$ and a possibly infinite family of $k$-uniform hypergraphs $\mathscr{P}$ we say a given hypergraph $H^{(k)}$* ***is $\eta$-far from $\mathscr{P}$*** *if every hypergraph $G^{(k)}$ on the same vertex set $V(H^{(k)})$ with $|G^{(k)} \triangle H^{(k)}| \leq \eta\binom{|V(H^{(k)})|}{k}$ satisfies $G^{(k)} \notin \mathscr{P}$, where $G^{(k)} \triangle H^{(k)}$ denotes the symmetric difference of the edge sets of $G^{(k)}$ and $H^{(k)}$.*

The main objective of this paper is to prove the following.

**Theorem 6.** *For every (possibly infinite) family $\mathscr{F}$ of $k$-uniform hypergraphs and every $\eta > 0$ there exist constants $c > 0$, $C > 0$, and $n_0$ such that the following holds.*

*Suppose $H^{(k)}$ is a $k$-uniform hypergraph on $n \geq n_0$ vertices. If for every $\ell = 1, \dots, C$ and every $F^{(k)} \in \mathscr{F}$ on $\ell$ vertices, $H^{(k)}$ contains at most $cn^\ell$ induced copies of $F^{(k)}$, then $H^{(k)}$ is not $\eta$-far from $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$.*

*In other words one can change (add/delete) up to at most $\eta\binom{n}{k}$ $k$-tuples in $V(H^{(k)})$ (to/from $H^{(k)}$) so that the resulting hypergraph $G^{(k)}$ contains no induced copy of any member of $\mathscr{F}$, i.e., so that $G^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$.*

*Moreover, since $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ is a subset of the family $\overline{\mathscr{F}}$ of all hypergraphs not contained in $\mathscr{F}$, such a hypergraph $H^{(k)}$ is also not $\eta$-far from $\overline{\mathscr{F}}$.*

For graphs Theorem 6 was first obtained by Alon and Shapira [2]. The proof in [2] is again based on the strong version of Szemerédi's regularity lemma from [1]. Another proof for graphs was found by Lovász and B. Szegedy [19] (see also [6]). Below we discuss a few consequences of Theorem 6, which motivated the original work for graphs.

Recall that for every hereditary property $\mathscr{P}$ of $k$-uniform hypergraphs, there exists a family of $k$-uniform hypergraphs $\mathscr{F}$ such that $\mathscr{P} = \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$. Consequently, Theorem 6 states that if $H^{(k)}$ is $\eta$-far from some hereditary property $\mathscr{P} = \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$, then it contains many ($cn^{|V(F^{(k)})|}$) induced copies of some "forbidden" hypergraph $F^{(k)} \in \mathscr{F}$ of size at most $C$, which "proves" that $H^{(k)}$ is not in $\mathscr{P}$. In other words, if $H^{(k)}$ is $\eta$-far from some given hereditary property $\mathscr{P}$, then it is "easy" to detect that $H^{(k)} \notin \mathscr{P}$. This implies Corollary 7, which we discuss after the following remark.

Note that if $\mathscr{P}$ is $\overline{\mathscr{F}}$, the complement of some family $\mathscr{F}$, then $\mathscr{P}$ is not necessarily hereditary. If $H^{(k)}$ is $\eta$-far from $\mathscr{P}$ in this case, then the "moreover-part" of Theorem 6 still implies that $H^{(k)}$ contains many induced copies of some forbidden hypergraph $F^{(k)} \in \mathscr{F}$ of bounded size. In this case, however, containing a forbidden hypergraph does not necessarily imply that $H^{(k)} \notin \mathscr{P}$. Hence, an analogous statement of Corollary 7 for arbitrary properties $\mathscr{P}$ (which is known to be false) is not implied.

Let us return to hereditary properties $\mathscr{P}$. For such properties Theorem 6 has an interesting consequence in the area of *property testing* (see, e.g., [14] for the definitions). We say a property $\mathscr{P}$ of hypergraphs (i.e., a family of hypergraphs) is *testable with one-sided error* if for every $\eta > 0$ there exists a constant $q = q(\mathscr{P}, \eta)$ and a randomized algorithm $\mathscr{A}$ which does the following: *For a given hypergraph $H^{(k)}$ the algorithm $\mathscr{A}$ can query some oracle whether a $k$-tuple $K$ of $V(H^{(k)})$ spans and edge in $H^{(k)}$ or not. After at most $q$ queries the algorithm outputs*

- *$H^{(k)} \in \mathscr{P}$ with probability 1 if $H^{(k)} \in \mathscr{P}$ and*
- *$H^{(k)} \notin \mathscr{P}$ with probability at least $2/3$ if $H^{(k)}$ is $\eta$-far from $\mathscr{P}$.*

*If $H^{(k)} \notin \mathscr{P}$ and $H^{(k)}$ is not $\eta$-far from $\mathscr{P}$, then there are no guarantees for the output of $\mathscr{A}$.*

Furthermore, we say a property $\mathscr{P}$ is *decidable* if there exists an algorithm which for every hypergraph $H^{(k)}$ distinguishes in finite time if $H^{(k)} \in \mathscr{P}$ or $H^{(k)} \notin \mathscr{P}$. In this context Theorem 6 implies the following.

**Corollary 7.** *Every decidable, hereditary property of hypergraphs is testable with one-sided error.*

The simple reduction of Corollary 7 from Theorem 6 in the context of graphs can be found in [2]. We omit it here, since the proof works verbatim for hypergraphs.

1.3. **Organization.** The proof of Theorem 6 combines ideas from the work of Alon, Fischer, Krivelevich, and M. Szegedy [1] and Lovász and B. Szegedy [19]. The main tool in the proof is the *hypergraph regularity lemma* from [25] (Theorem 20) and the accompanying *counting lemma* from [20] (Theorem 13). In Section 2 we introduce the necessary definitions to state Theorem 13 and Theorem 20.

Then in Section 3 we prove a few preparatory results for the proof of the main result, Theorem 6, which follows in Section 4.

## 2. Regularity method for hypergraphs

2.1. **Basic definitions.** For real constants $\alpha$, $\beta$, and a non-negative constants $\xi$ we sometimes write $\alpha = \beta \pm \xi$, if $\beta - \xi \leq \alpha \leq \beta + \xi$. For integers $\ell \geq j \geq 1$, the notation $[\ell]$ denotes the set of integers $\{1, \dots, \ell\}$ and $\binom{[\ell]}{j}$ denotes the set of all unordered $j$-tuples from $[\ell]$.

In this paper $\ell$-partite, $j$-uniform hypergraphs play a special rôle, where $j \leq \ell$. Given vertex sets $V_1, \dots, V_\ell$, we denote by $K_\ell^{(j)}(V_1, \dots, V_\ell)$ the **complete** $\ell$-partite, $j$-uniform hypergraph (i.e., the family of all $j$-element subsets $J \subseteq \bigcup_{i \in [\ell]} V_i$ satisfying $|V_i \cap J| \leq 1$ for every $i \in [\ell]$). If $|V_i| = m$ for every $i \in [\ell]$, then an $(m, \ell, j)$**-hypergraph** $H^{(j)}$ on $V_1 \cup \dots \cup V_\ell$ is any subset of $K_\ell^{(j)}(V_1, \dots, V_\ell)$. The vertex partition $V_1 \cup \dots \cup V_\ell$ is an $(m, \ell, 1)$-hypergraph $H^{(1)}$. (This definition may seem artificial right now, but it will simplify later notation.) For $j \leq i \leq \ell$ and set $\Lambda_i \in \binom{[\ell]}{i}$, we denote by $H^{(j)}[\Lambda_i] = H^{(j)}\big[\bigcup_{\lambda \in \Lambda_i} V_\lambda\big]$ the sub-hypergraph of the $(m, \ell, j)$-hypergraph $H^{(j)}$ induced on $\bigcup_{\lambda \in \Lambda_i} V_\lambda$.

For an $(m, \ell, j)$-hypergraph $H^{(j)}$ and an integer $2 \leq j \leq i \leq \ell$, we denote by $\mathcal{K}_i(H^{(j)})$ the family of all $i$-element subsets of $V(H^{(j)})$ which span complete sub-hypergraphs in $H^{(j)}$ of order $i$. For $1 \leq i \leq \ell$, we denote by $\mathcal{K}_i(H^{(1)})$ the family of all $i$-element subsets of $V(H^{(1)})$ which 'cross' the partition $V_1 \cup \dots \cup V_\ell$, i.e., $I \in \mathcal{K}_i(H^{(1)})$ if, and only if, $|I \cap V_s| \leq 1$ for all $1 \leq s \leq \ell$. For $2 \leq j \leq i \leq \ell$,

$|\mathcal{K}_i(H^{(j)})|$ is the number of all copies of $K_i^{(j)}$ in $H^{(j)}$. Given an $(m,\ell,j-1)$-hypergraph $H^{(j-1)}$ and an $(m,\ell,j)$-hypergraph $H^{(j)}$, we say $H^{(j-1)}$ **underlies** $H^{(j)}$ if $H^{(j)} \subseteq \mathcal{K}_j(H^{(j-1)})$. This brings us to one of the main concepts of this paper, the notion of a complex.

**Definition 8** ($(m,\ell,h)$**-complex**). *Let $m \geq 1$ and $\ell \geq h \geq 1$ be integers. An $(m,\ell,h)$-**complex** $\boldsymbol{H}$ is a collection of $(m,\ell,j)$-hypergraphs $\{H^{(j)}\}_{j=1}^h$ such that*

(*a*) *$H^{(1)}$ is an $(m,\ell,1)$-hypergraph, i.e., $H^{(1)} = V_1 \cup \dots \cup V_\ell$ with $|V_i| = m$ for $i \in [\ell]$, and*
(*b*) *$H^{(j-1)}$ underlies $H^{(j)}$ for $2 \leq j \leq h$, i.e., $H^{(j)} \subseteq \mathcal{K}_j(H^{(j-1)})$.*

We sometimes shorten the terminology $(m,\ell,h)$-hypergraph and $(m,\ell,h)$-complex to $(\ell,h)$-*hypergraph* and $(\ell,h)$-*complex*, when the cardinality $m = |V_1| = \dots = |V_s|$ isn't of primary concern.

2.2. **Regular complexes.** We begin by defining a relative density of a $j$-uniform hypergraph w.r.t. $(j-1)$-uniform hypergraph on the same vertex set.

**Definition 9** (**relative density**). *Let $H^{(j)}$ be a $j$-uniform hypergraph and let $H^{(j-1)}$ be a $(j-1)$-uniform hypergraph on the same vertex set. We define the* ***density of*** $H^{(j)}$ **w.r.t.** $H^{(j-1)}$ *as*

$$d\big(H^{(j)}\big|H^{(j-1)}\big) = \begin{cases} \frac{|H^{(j)} \cap \mathcal{K}_j(H^{(j-1)})|}{|\mathcal{K}_j(H^{(j-1)})|} & \text{if} \quad \big|\mathcal{K}_j(H^{(j-1)})\big| > 0 \\ 0 & \text{otherwise}. \end{cases}$$

We also define a notion of regularity for $(m,j,j)$-hypergraphs w.r.t. some underlying $(m,j,j-1)$-hypergraphs.

**Definition 10.** *Let constants $\delta > 0$ and $d \geq 0$ and a positive integer $r$ be given along with an $(m,j,j)$-hypergraph $H^{(j)}$ and an $(m,j,j-1)$-hypergraph $H^{(j-1)}$ on the same vertex set. We say $H^{(j)}$ is $(\delta,d,r)$-**regular** w.r.t. $H^{(j-1)}$ if for every collection $\boldsymbol{X} = \{X_1^{(j-1)},\dots,X_r^{(j-1)}\}$ of not necessarily disjoint sub-hypergraphs of $H^{(j-1)}$ satisfying*

$$\Big|\bigcup_{i\in[r]} \mathcal{K}_j(X_i^{(j-1)})\Big| > \delta \Big|\mathcal{K}_j(H^{(j-1)})\Big|\,, \quad \text{we have} \quad d\big(H^{(j)}\big|\boldsymbol{X}\big) = d \pm \delta\,,$$

*where*

$$d\big(H^{(j)}\big|\boldsymbol{X}\big) = \frac{\big|H^{(j)} \cap \bigcup_{i\in[r]} \mathcal{K}_j(X_i^{(j-1)})\big|}{\big|\bigcup_{i\in[r]} \mathcal{K}_j(X_i^{(j-1)})\big|}\,.$$

*We also write $(\delta,*,r)$-**regular** to mean $\big(\delta, d\big(H^{(k)}\big|H^{(k-1)}\big), r\big)$-regular. Moreover, we say $H^{(j)}$ is $(\delta,\geq d,r)$-**regular** w.r.t. $H^{(j-1)}$ if $d\big(H^{(k)}\big|H^{(k-1)}\big) \geq d$ and $H^{(j)}$ is $(\delta,*,r)$-regular w.r.t. $H^{(j-1)}$.*

We extend the notion of regular $(m,j,j)$-hypergraph to $(m,\ell,j)$-hypergraphs.

**Definition 11** ($(\delta,d,r)$**-regular**). *For positive integers $m$, $\ell \geq j$ we say an $(m,\ell,j)$-hypergraph $H^{(j)}$ is $(\delta,d,r)$-**regular** (resp. $(\delta,\geq d,r)$-**regular**) w.r.t. an $(m,\ell,j-1)$-hypergraph $H^{(j-1)}$ if for every $\Lambda_j \in \binom{\ell}{j}$, the restriction $H^{(j)}[\Lambda_j] = H^{(j)}\big[\bigcup_{\lambda\in\Lambda_j} V_\lambda\big]$ is $(\delta,d,r)$-regular (resp. $(\delta,\geq d,r)$-regular) w.r.t. the restriction $H^{(j-1)}[\Lambda_j] = H^{(j-1)}\big[\bigcup_{\lambda\in\Lambda_j} V_\lambda\big]$.*

We now extend the notion of regularity from hypergraphs to complexes.

**Definition 12** ($(\boldsymbol{\delta},\boldsymbol{d},r)$**-regular complex**). *Let $\boldsymbol{\delta} = (\delta_2,\dots,\delta_h)$ be a vector of positive reals and let $\boldsymbol{d} = (d_2,\dots,d_h)$ be a vector of non-negative reals. We say an $(m,\ell,h)$-complex $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^h$ is $(\boldsymbol{\delta},\boldsymbol{d},r)$-**regular** (resp. $(\boldsymbol{\delta},\geq\boldsymbol{d},r)$-**regular**) if*

(*i*) $H^{(2)}$ *is* $(\delta_2, d_2, 1)$*-regular (resp.* $(\delta_2, \geq d_2, 1)$*-regular) w.r.t.* $H^{(1)}$ *and*
(*ii*) $H^{(j)}$ *is* $(\delta_j, d_j, r)$*-regular (resp.* $(\delta_j, \geq d_j, 1)$*-regular) w.r.t.* $H^{(j-1)}$ *for every* $j = 3, \dots, h$.

2.3. **Counting lemma.** The following theorem was proved by Nagle and authors in [20]. It was one of the key ingredients for the proof of the removal lemma, Theorem 2, and will also play a crucial rôle here.

**Theorem 13** (Counting lemma [20, Corollary 15])**.** *For all integers* $2 \leq k \leq \ell$ *the following is true:* $\forall \gamma > 0\ \forall d_k > 0\ \exists \delta_k > 0\ \forall d_{k-1} > 0\ \exists \delta_{k-1} > 0\ \dots\ \forall d_2 > 0\ \exists \delta_2 > 0$ *and there are integers* $r$ *and* $m_0$ *so that, with* $\boldsymbol{d} = (d_2, \dots, d_k)$ *and* $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ *and* $m \geq m_0$*, whenever* $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^{k}$ *is a* $(\boldsymbol{\delta}, \geq \boldsymbol{d}, r)$*-regular* $(m, \ell, k)$*-complex, then*

$$\left|\mathcal{K}_\ell(H^{(k)})\right| \geq (1-\gamma) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \times m^\ell .$$

□

Since Theorem 6 concerns induced copies of hypergraphs an induced version of the counting lemma, which is a simple corollary of Theorem 13, will be useful. For the statement of that version we need the following definition.

**Definition 14** (($\boldsymbol{\delta}, \boldsymbol{d}, r$)**-regular, induced** ($m, F^{(k)}$)**-complex**)**.** *Let* $F^{(k)}$ *be a* $k$*-uniform hypergraph with* $V(F^{(k)}) = [\ell]$*. Let* $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ *be a vector of positive reals and let* $\boldsymbol{d} = (d_2, \dots, d_k)$ *be a vector of non-negative reals. We say an* $(m, \ell, k)$*-complex* $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^{k}$ *with vertex partition* $V_1 \cup \dots \cup V_k$ *is a* $(\boldsymbol{\delta}, \geq \boldsymbol{d}, r)$***-regular, induced*** $(m, F^{(k)})$***-complex*** *if*

(*i*) *the complex* $\{H^{(j)}\}_{j=1}^{k-1}$ *is a* $(\boldsymbol{\delta}', \geq \boldsymbol{d}', r)$*-regular* $(m, \ell, k-1)$*-complex with* $\boldsymbol{\delta}' = (\delta_2, \dots, \delta_{k-1})$ *and* $\boldsymbol{d}' = (d_2, \dots, d_{k-1})$*,*
(*ii*) *for every* $k$*-tuple in* $K = \{\lambda_1, \dots, \lambda_k\} \in \binom{\ell}{k}$ *we have*
  (*a*) *if* $K$ *is an edge in* $F^{(k)}$*, then the* $(m, k, k)$*-hypergraph* $H^{(k)}[K] = H^{(k)}[\bigcup_{j=1}^{k} V_{\lambda_j}]$ *is* $(\delta_k, \geq d_k, r)$*-regular w.r.t.* $H^{(k-1)}[K]$
  (*b*) *if* $K$ *is not an edge in* $F^{(k)}$*, then the* $(m, k, k)$*-hypergraph complement* $\mathcal{K}_k(H^{(k-1)}[K]) \setminus H^{(k)}[K]$ *is* $(\delta_k, \geq d_k, r)$*-regular w.r.t.* $H^{(k-1)}[K]$*.*

We then state the induced version of Theorem 13.

**Corollary 15.** *For all integers* $2 \leq k \leq \ell$ *the following is true:* $\forall \gamma > 0\ \forall d_k > 0\ \exists \delta_k > 0\ \forall d_{k-1} > 0\ \exists \delta_{k-1} > 0\ \dots\ \forall d_2 > 0\ \exists \delta_2 > 0$ *and there are integers* $r$ *and* $m_0$ *so that for every* $m \geq m_0$ *and for every* $k$*-uniform hypergraph* $F^{(k)}$ *with vertex set* $[\ell]$ *the following holds.*

*Let* $\boldsymbol{d} = (d_2, \dots, d_k)$*,* $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$*, and let* $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^{k}$ *be a* $(\boldsymbol{\delta}, \geq \boldsymbol{d}, r)$*-regular, induced* $(m, F^{(k)})$*-complex with vertex partition* $V_1 \cup \dots \cup V_k$*. Then* $H^{(k)}$ *contains at least*

$$(1-\gamma) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \times m^\ell$$

*induced copies of* $F^{(k)}$*.*

*Proof.* It follows from Definition 14, that if $\boldsymbol{H} = \{H^{(j)}\}_{j=1}^{k}$ is a $(\boldsymbol{\delta}, \geq \boldsymbol{d}, r)$-regular, induced $(m, F^{(k)})$-complex, then $\tilde{\boldsymbol{H}} = \{H^{(1)}, \dots, H^{(k-1)}, \tilde{H}^{(k)}\}$ is a $(\boldsymbol{\delta}, \geq \boldsymbol{d}, r)$-regular $(m, \ell, k)$-complex, where $\tilde{H}^{(k)}$ is defined by setting for every $K \in \binom{\ell}{k}$

$$\tilde{H}^{(k)}[K] = \begin{cases} H^{(k)}[K] & \text{if} \quad K \in F^{(k)}, \\ \mathcal{K}_k(H^{(k-1)}[K]) \setminus H^{(k)}[K] & \text{if} \quad K \notin F^{(k)}. \end{cases}$$

Moreover, every clique $K_k^{(\ell)}$ in $\tilde{H}^{(k)}$ corresponds to an induced copy of $F^{(k)}$ in $H^{(k)}$ and, hence, Corollary 15 follows from Theorem 13 applied to $\tilde{\boldsymbol{H}}$. $\square$

2.4. **Regularity lemma.** In this section we introduce some more notation needed for the statement of the hypergraph regularity lemma, Theorem 20, from [25]. First we define the refinement of a partition.

**Definition 16** (**refinement**). *Suppose $A \supseteq B$ are sets, $\mathscr{A}$ is a partition of $A$, and $\mathscr{B}$ is a partition of $B$. We say $\mathscr{A}$* ***refines*** *$\mathscr{B}$ and write $\mathscr{A} \prec \mathscr{B}$ if for every $\mathcal{A} \in \mathscr{A}$ there either exist a $\mathcal{B} \in \mathscr{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$ or $\mathcal{A} \subseteq A \setminus B$.*

2.4.1. *Partitions.* The regularity lemma for $k$-uniform hypergraphs provides a well-structured family of partitions $\mathscr{P} = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ of vertices, pairs, ..., and $(k-1)$-tuples of the vertex set. We now discuss the structure of these partitions recursively. Here the partition classes of $\mathscr{P}^{(j)}$ will be $(j,j)$-hypergraphs, i.e., $j$-uniform, $j$-partite hypergraphs.

Let $k$ be a fixed integer and $V$ be a set of vertices. Let $\mathscr{P}^{(1)} = \{V_1, \dots, V_{|\mathscr{P}^{(1)}|}\}$ be a partition of $V$. For every $1 \le j \le |\mathscr{P}^{(1)}|$, let $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ be the family of all crossing $j$-tuples $J$, i.e., the set of $j$-tuples which satisfy $|J \cap V_i| \le 1$ for every $V_i \in \mathscr{P}^{(1)}$.

Suppose for each $1 \le i \le j-1$ partitions $\mathscr{P}^{(i)}$ of $\mathrm{Cross}_i(\mathscr{P}^{(1)})$ into $(i,i)$-hypergraphs are given. Then for every $(j-1)$-tuple $I$ in $\mathrm{Cross}_{j-1}(\mathscr{P}^{(1)})$, there exists a unique $(j-1,j-1)$-hypergraph $P^{(j-1)} = P^{(j-1)}(I) \in \mathscr{P}^{(j-1)}$ so that $I \in P^{(j-1)}$. For every $j$-tuple $J$ in $\mathrm{Cross}_j(\mathscr{P}^{(1)})$, we define the **polyad** of $J$

$$\hat{P}^{(j-1)}(J) = \bigcup \left\{ P^{(j-1)}(I)\colon\ I \in \tbinom{J}{j-1} \right\}.$$

In other words, $\hat{P}^{(j-1)}(J)$ is the unique set of $j$ partition classes (or $(j-1,j-1)$-hypergraphs) of $\mathscr{P}^{(j-1)}$ each containing a $(j-1)$-subset of $J$. Observe that $\hat{P}^{(j-1)}(J)$ we view as a $(j,j-1)$-hypergraph. More generally, for $1 \le i < j$, we set

$$\hat{P}^{(i)}(J) = \bigcup \left\{ P^{(i)}(I)\colon\ I \in \tbinom{J}{i} \right\} \qquad \text{and} \qquad \boldsymbol{P}(J) = \left\{ \hat{P}^{(i)}(J) \right\}_{i=1}^{j-1}. \tag{1}$$

Next, we define $\hat{\mathscr{P}}^{(j-1)}$, the family of all polyads

$$\hat{\mathscr{P}}^{(j-1)} = \left\{ \hat{P}^{(j-1)}(J)\colon\ J \in \mathrm{Cross}_j(\mathscr{P}^{(1)}) \right\}.$$

Note that $\hat{P}^{(j-1)}(J)$ and $\hat{P}^{(j-1)}(J')$ are not necessarily distinct for different $j$-tuples $J$ and $J'$. We view $\hat{\mathscr{P}}^{(j-1)}$ as a set and, consequently, $\{\mathcal{K}_j(\hat{P}^{(j-1)})\colon\ \hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ is a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$.

The structural requirement on the partition $\mathscr{P}^{(j)}$ of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ is

$$\mathscr{P}^{(j)} \prec \{\mathcal{K}_j(\hat{P}^{(j-1)})\colon\ \hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}. \tag{2}$$

In other words, we require that the set of cliques spanned by a polyad in $\hat{\mathscr{P}}^{(j-1)}$ is sub-partitioned in $\mathscr{P}^{(j)}$ and every partition class in $\mathscr{P}^{(j)}$ belongs to precisely one polyad in $\hat{\mathscr{P}}^{(j-1)}$. Note that (2) implies (inductively) that $\boldsymbol{P}(J)$ defined in (1) is a $(j,j-1)$-complex. On a related note, we shall often drop the argument $J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})$ from the notation $\hat{P}^{(j-1)}(J)$.

Throughout this paper, we want to have an upper bound on the number of partition classes in $\mathscr{P}^{(j)}$, and more specifically, over the number of classes contained in $\mathcal{K}_j(\hat{P}^{(j-1)})$ for a fixed polyad $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$. We make this precise in the following definition.

**Definition 17** (**family of partitions**). *Suppose $V$ is a set of vertices, $k \ge 2$ is an integer and $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ is a vector of positive integers. We say $\mathscr{P} =$*

$\mathscr{P}(k-1,\boldsymbol{a}) = \{\mathscr{P}^{(1)},\dots,\mathscr{P}^{(k-1)}\}$ ***is a family of partitions on*** $V$, *if it satisfies the following:*

($i$) $\mathscr{P}^{(1)}$ *is a partition of* $V$ *into* $a_1$ *classes,*
($ii$) $\mathscr{P}^{(j)}$ *is a partition of* $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ *satisfying:*

$$\mathscr{P}^{(j)} \prec \{\mathcal{K}_j(\hat{P}^{(j-1)})\colon\ \hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$$
$$\text{and}\quad \left|\left\{P^{(j)} \in \mathscr{P}^{(j)}\colon\ P^{(j)} \subseteq \mathcal{K}_j(\hat{P}^{(j-1)})\right\}\right| = a_j \quad \text{for every}\quad \hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\,.$$

*Moreover, we say* $\mathscr{P} = \mathscr{P}(k-1,\boldsymbol{a})$ *is* $T$*-bounded, if* $\max\{a_1,\dots,a_{k-1}\} \le T$.

It is easy to see that for a $T$-bounded family of partitions $\mathscr{P}(k-1,\boldsymbol{a})$ and an integer $j$, $2 \le j \le k-1$, we have

$$|\hat{\mathscr{P}}^{(j-1)}| = \binom{a_1}{j}\prod_{h=2}^{j-1} a_h^{\binom{j}{h}} \le T^{2^k}. \tag{3}$$

2.4.2. *Regular partitions.* The following two definitions describe the "regularity" properties of the partition the regularity lemma shall provide. While the first definition deals with regularity properties of the auxiliary structure, the second definition describes how $H^{(k)}$ interacts with the partition.

**Definition 18** (**(**$(\mu,\boldsymbol{\delta},\boldsymbol{d},r)$**-equitable**)**.** *Suppose* $V$ *is a set of* $n$ *vertices,* $\mu > 0$, $\boldsymbol{\delta} = (\delta_2,\dots,\delta_{k-1}) \in (0,1]^{k-2}$ *and* $\boldsymbol{d} = (d_2,\dots,d_{k-1}) \in [0,1]^{k-2}$ *are vectors of reals and* $r$ *is a positive integer.*

*We say a family of partitions* $\mathscr{P} = \mathscr{P}(k-1,\boldsymbol{a})$ *on* $V$ *is* $(\mu,\boldsymbol{\delta},\boldsymbol{d},r)$***-equitable*** *if:*

($a$) $\left|\binom{V}{k} \setminus \mathrm{Cross}_k(\mathscr{P}^{(1)})\right| \le \mu\binom{n}{k}$,
($b$) $\mathscr{P}^{(1)} = \{V_i\colon\ i \in [a_1]\}$ *is an equitable vertex partition, i.e.,* $|V_1| \le \dots \le |V_{a_1}| \le |V_1|+1$, *and*
($c$) *for all but* $\mu\binom{n}{k}$ $k$*-tuples* $K \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$ *the complex* $\boldsymbol{P}(K)$ *(see (1)) is a* $(\boldsymbol{\delta},\boldsymbol{d},r)$*-regular* $(n/a_1,k,k-1)$*-complex.**

The following definition describes the "relation" of the $k$-uniform hypergraph and the partition provided by the regularity lemma.

**Definition 19** (**(**$(\delta_k,*,r)$**-regular w.r.t.** $\mathscr{P}$)**.** *Suppose* $\delta_k > 0$ *and* $r$ *is a positive integer. Let* $H^{(k)}$ *be a* $k$*-uniform hypergraph with vertex set* $V$ *and* $\mathscr{P} = \mathscr{P}(k-1,\boldsymbol{a})$ *be a family of partitions on* $V$. *We say* $H^{(k)}$ is $(\delta_k,*,r)$-regular w.r.t. $\mathscr{P}$, *if*

$$\Big|\bigcup\Big\{\mathcal{K}_k(\hat{P}^{(k-1)})\colon\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$$
$$\text{and } H^{(k)} \text{ is not } (\delta_k,*,r)\text{-regular w.r.t. } \hat{P}^{(k-1)}\Big\}\Big| \le \delta_k\binom{|V|}{k}.$$

Finally we state the regularity lemma for hypergraphs.

**Theorem 20** (Regularity lemma)**.** *Let* $k \ge 2$ *be a fixed integer. For every positive integer* $S$, *all positive constants* $\mu$ *and* $\delta_k$ *and functions* $\delta_j\colon (0,1]^{k-j} \to (0,1]$ *for* $j = 2,\dots k-1$ *and* $r\colon \mathbb{N}\times(0,1]^{k-2} \to \mathbb{N}$ *there are integers* $T_0$ *and* $n_0$ *and* $d_0 > 0$ *so that the following holds.*

*For every* $k$*-uniform hypergraph* $H^{(k)}$ *satisfying* $|V(H^{(k)})| = n \ge n_0$ *and every* $S$*-bounded family of partitions* $\mathscr{Q} = \mathscr{Q}(k-1,\boldsymbol{a}^{\mathscr{Q}})$ *with an equitable vertex partition, i.e.,* $\mathscr{Q}^{(1)} = \{V_1,\dots,V_{a_1^{\mathscr{Q}}}\}$ *satisfies* $|V_1| \le \dots \le |V_{a_1}| \le |V_1|+1$, *there exists a*

*Strictly speaking in view of ($b$) the vertex classes of $\mathscr{P}^{(1)}$ have sizes in $\{\lfloor n/a_1\rfloor, \lceil n/a_1\rceil\}$. We, however, omit floors and ceilings, as they have no influence on our arguments.

*family of partitions* $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ *and a vector* $\boldsymbol{d} = (d_2, \dots, d_{k-1}) \in (0,1]^{k-2}$ *so that for*

$$\boldsymbol{\delta} = \boldsymbol{\delta}(\boldsymbol{d}) = (\delta_2, \dots, \delta_{k-1}), \quad \textit{where} \quad \delta_j = \delta_j(d_j, \dots, d_{k-1}) \quad \textit{for } j = 2, \dots, k-1,$$
$$\textit{and} \quad r = r(a_1^{\mathscr{P}}, d_2, \dots, d_{k-1})$$

*the following holds:*

(*i*) $\mathscr{P}$ *is* $(\mu, \boldsymbol{\delta}, \boldsymbol{d}, r)$*-equitable and* $T_0$*-bounded,*
(*ii*) $H^{(k)}$ *is* $(\delta_k, *, r)$*-regular w.r.t.* $\mathscr{P}$*,*
(*iii*) $\mathscr{P} \prec \mathscr{Q}$*, i.e.,* $\mathscr{P}^{(j)} \prec \mathscr{Q}^{(j)}$ *for every* $j = 1, \dots, k-1$*, and*
(*iv*) $d_j \geq d_0$ *for every* $j = 2, \dots, k-1$*.*

□

Theorem 20 slightly differs from the hypergraph regularity lemma of Rödl and Skokan from [25]. However, a proof of Theorem 20 follows along the lines of [25]. We discuss the the five small differences below.

- In the definition of family of partitions (Definition 17), we require that for every $j = 2, \dots, k-1$ and every $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$ there are *precisely* $a_j$ partition classes in $\mathscr{P}^{(j)}$, which decompose $\mathcal{K}_j(\hat{P}^{(k-1)})$. In [25] $a_j$ is only an upper bound of the number of partition classes contained in $\mathcal{K}_j(\hat{P}^{(k-1)})$. We may think of simply adding some artificial empty classes to $\mathscr{P}^{(j)}$ to have precisely $a_j$ classes for every $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$.
- By Definition 18 part (*b*) we require that the vertex classes of $\mathscr{P}^{(1)}$ differ in size by at most 1. We can require this additional assertion, provided the initial vertex partition of $\mathscr{Q}$ has the same property, since it is well know that such an assertion holds for the graph regularity lemma of Szemerédi [32] and since the hypergraph regularity lemma in [25] is proved by induction on the uniformity. For more details we refer to [25, Remark 7.19].
- We also use a slightly different notation for the boundedness of a partition. More precisely the lemma in [25] admits a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ such that $|\hat{\mathscr{P}}^{(k-1)}| \leq T_0$. However, this clearly implies by Definition 17 that $\max_{j \in [k-1]} a_j \leq T_0$, i.e., $\mathscr{P}$ is $T_0$-bounded as stated in (*i*) of Theorem 20.
- Another difference concerns assertion (*iii*) in Theorem 20. Recall that the proof of Szemerédi's regularity lemma relies on a procedure in which a given non-regular vertex partition $V_0 \cup V_1 \cup \dots \cup V_s$ will be "almost" refined by a partition $W_0 \cup W_1 \cup \dots \cup W_t$. Here "almost" refinement means that only the "exceptional" class $W_0$ may not be contained in $V_0$, while for every other class $W_j$ there exist some $V_i \supseteq W_j$. However the initial vertex partition $U_1 \cup \dots \cup U_r$ is completely arbitrary and one can insist that the partitions obtained in the proof always refine the initial one, if one allows not only one "exceptional" class, but one exceptional class, say $U_{i,0} \subseteq U_i$, for each $i \in [r]$, i.e., one exceptional class for every vertex class from the initial partition.

  Similar adjustments can be made in the proof of the hypergraph regularity lemma from [25], this way we will have for every $j = 1, \dots, k-1$ and every $Q^{(j)} \in \mathscr{Q}^{(j)}$ (of the given partition) always precisely one exceptional class $Q_0^{(j)}$.

  We also note that such an argument was carried out in [25, Corollary 12.1], where the additional assertion (*iii*) of Theorem 20 was proved in the similar case when $\mathscr{Q}$ is replaced by an $(\ell, k-1)$-complex $\boldsymbol{G} = \{G^{(j)}\}_{j=1}^{k-1}$ and "refinement" means for every $j = 1, \dots, k-1$ and every $P^{(j)}$ either

$P^{(j)} \subseteq G^{(j)}$ or $P^{(j)} \cap G^{(j)} = \emptyset$. The proof for a bounded partition $\mathscr{Q}$ instead of a complex $\boldsymbol{G}$ is the same and follows the lines of the proof of [25, Corollary 12.1].

- The last difference concerns (*iv*). This condition was not "built in" the regularity lemma of [25], but was explicitly proved, e.g., in [26, Claim 6.1]. We outline the simple proof here.

  First recall that by Definition 18 the number of non-crossing $k$-tuples, as well as, the number of $k$-tuples in irregular polyads is bounded by $\mu\binom{n}{k}$ for each reason. Therefore if $\mu < 1/8$ (an assumption one can clearly make without loss of generality) there are at least $(1-2\mu)\binom{n}{k} > (3/4)\binom{n}{k}$ $k$-tuples in regular polyads. Now all those $k$-tuples have its $\binom{k}{j}$ $j$-tuples ($2 \leq j < k$) in $(d_j, \delta_j, r)$-regular $(j,j)$-hypergraphs from $\mathscr{P}^{(j)}$. Since the number of such hypergraphs is bounded by $T_0^{2^j} \leq T_0^{2^k}$ we infer by the $(d_j, \delta_j, r)$-regularity that $T_0^{2^k}(d_j + \delta_j)\binom{n}{k} \geq \frac{3}{4}\binom{n}{k}$, which provided $\delta_j(d_j, \dots, d_{k-1}) \leq d_j/2$ (an assumption one can clearly make without loss of generality) implies $d_j \geq 1/(2T_0^{2^k}) =: d_0$.

## 3. Auxiliary lemmas

3.1. **Cluster hypergraphs.** An important part of the argument in the proof of Theorem 6 will be to compare hypergraphs of very different sizes to find two of "similar structure." For that we will use the hypergraph regularity lemma. Suppose hypergraphs of different size were regularized by Theorem 20 with the *same* input parameters. Then sizes of all of the families of partitions corresponding to each of the hypergraphs are bounded by the same $T_0$. Let us assume for now that all the partitions have the same size or more precisely have the same vector $\boldsymbol{a}$. Then we would like to say that two hypergraphs have the same structure, if there densities are similar on "every pair of corresponding polyads," for an appropriate bijection between the polyads of two partitions.

The similar idea of comparing "cluster graphs" corresponding to graphs of various sizes was used by Lovász and B. Szegedy [19]. The structure of partition yielded by the hypergraph regularity lemma is unfortunately more complicated than that for Szemerédi's regularity lemma. In Section 3.1.1 we first introduce the notion of a *labeled* family of partitions, which in the graph case corresponds to a labeling of the vertex classes of the regular partition. Then, in Section 3.1.2, we develop the notion, which will later allow us to identify hypergraphs of the same structure, which is similar to the edge weights of the cluster graph.

3.1.1. *Labeled partitions.* In this paper it will be convenient to consider labeled families of partitions. Let $\mathscr{P}(k-1, \boldsymbol{a})$ be a family of partitions on $V$ (see Definition 17). Consider an arbitrary numbering of the vertex classes of $\mathscr{P}^{(1)}$, i.e., $\mathscr{P}^{(1)} = \{V_i \colon i \in [a_1]\}$. For $j = 2, \dots, k-1$ let $\varphi^{(j)} \colon \mathscr{P}^{(j)} \to [a_j]$ be a labeling such that for every polyad $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$ the members of $\{P^{(j)} \in \mathscr{P}^{(j)} \colon P^{(j)} \subseteq \mathcal{K}_j(\hat{P}^{(j-1)})\}$ are numbered from 1 to $a_j$ in an arbitrary way.

This way, we obtain for every $k$-tuple $K = \{v_1, \dots, v_k\} \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$ an integer vector $\hat{\boldsymbol{x}}_K = (\boldsymbol{x}_K^{(1)}, \dots, \boldsymbol{x}_K^{(k-1)})$, where

$$\boldsymbol{x}_K^{(1)} = (\alpha_1 < \dots < \alpha_k) \text{ so that w.l.o.g. } K \cap V_{\alpha_i} = \{v_i\} \tag{4}$$

and for $j = 2, \dots, k-1$ we set

$$\boldsymbol{x}_K^{(j)} = \Big(\varphi^{(j)}(P^{(j)}) \colon \{v_\lambda \colon \lambda \in \Lambda\} \in P^{(j)}\Big)_{\Lambda \in \binom{[k]}{j}} \tag{5}$$

Let $\binom{[a_1]}{k}_< = \{(\alpha_1,\dots,\alpha_k)\colon 1\le \alpha_1<\dots<\alpha_k\le a_1\}$ be the set of all "naturally" ordered $k$-element subsets of $[a_1]$ and set

$$\hat{A}(k-1,\boldsymbol{a}) = \binom{[a_1]}{k}_< \times \prod_{j=2}^{k-1} \underbrace{[a_j]\times\dots\times[a_j]}_{\binom{k}{j}\text{-times}} \tag{6}$$

for the address space of all $k$-tuples $K\in\mathrm{Cross}_k(\mathscr{P}^{(1)})$. The definitions above yield $\hat{\boldsymbol{x}}_K\in\hat{A}(k-1,\boldsymbol{a})$ for every $K\in\mathrm{Cross}_k(\mathscr{P}^{(1)})$. Moreover, for every $\hat{P}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}$ we have

$$\hat{\boldsymbol{x}}_K=\hat{\boldsymbol{x}}_{K'} \text{ for all } K,K'\in\mathcal{K}_k(\hat{P}^{(k-1)}) \tag{7}$$

hence, for every $\hat{P}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}$ with $\mathcal{K}_k(\hat{\mathscr{P}}^{(k-1)})\ne\emptyset$ we may set

$$\hat{\boldsymbol{x}}(\hat{P}^{(k-1)})=\hat{\boldsymbol{x}}_K \text{ for some } K\in\mathcal{K}_k(\hat{P}^{(k-1)})\,. \tag{8}$$

Let

$$\hat{\mathscr{P}}^{(k-1)}_{\ne\emptyset}=\big\{\hat{P}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}\colon\ \mathcal{K}_k(\hat{\mathscr{P}}^{(k-1)})\ne\emptyset\big\}$$

and

$$\hat{A}_{\ne\emptyset}=\big\{\hat{\boldsymbol{x}}\in\hat{A}(k-1,\boldsymbol{a})\colon\ \exists\ \hat{P}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\ne\emptyset} \text{ such that } \hat{\boldsymbol{x}}(\hat{P}^{(k-1)})=\hat{\boldsymbol{x}}\big\}\,.$$

It is easy to see that the definition in (8) establishes a bijection between $\hat{\mathscr{P}}^{(k-1)}_{\ne\emptyset}$ and $\hat{A}_{\ne\emptyset}$.

Moreover, since $|\hat{\mathscr{P}}^{(k-1)}|=|\hat{A}(k-1,\boldsymbol{a})|$ (see (3) and (6)) this bijection can be extended to a bijection between $\hat{\mathscr{P}}^{(k-1)}$ and $\hat{A}(k-1,\boldsymbol{a})$. The inverse bijection maps $\hat{\boldsymbol{x}}\mapsto\hat{P}^{(k-1)}(\hat{\boldsymbol{x}})$ and in the case $\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\boldsymbol{x}}))\ne\emptyset$, i.e., $\hat{\boldsymbol{x}}\in\hat{A}_{\ne\emptyset}$ then

$$\boldsymbol{P}(\hat{\boldsymbol{x}})=\boldsymbol{P}(K) \text{ for some } K\in\hat{P}^{(k-1)}(\hat{\boldsymbol{x}})\,,$$

is well defined due to (7). Note that $\boldsymbol{P}(\hat{\boldsymbol{x}})=\{P^{(j)}\}_{j=1}^{k-1}$ is a $(k,k-1)$-complex with $P^{(k-1)}=\hat{P}^{(k-1)}(\hat{\boldsymbol{x}})$. For later reference we summarize the above.

**Definition 21** (**labeled family of partitions**). *Suppose $k\ge2$ is an integer and $\boldsymbol{a}=(a_1,\dots,a_{k-1})$ is a vector of positive integers. We say*

$$\hat{A}(k-1,\boldsymbol{a}) = \binom{[a_1]}{k}_< \times \prod_{j=2}^{k-1} \underbrace{[a_j]\times\dots\times[a_j]}_{\binom{k}{j}\text{-times}},$$

*is the **address space**.*

*For a family of partitions $\mathscr{P}(k-1,\boldsymbol{a})$ on some vertex set $V=V_1\cup\dots\cup V_{a_1}$ we say a set of mappings $\boldsymbol{\varphi}=\{\varphi^{(2)},\dots,\varphi^{(k-1)}\}$, $\varphi^{(j)}\colon\mathscr{P}^{(j)}\to[a_j]$ for every $j=2,\dots,k-1$ is an **$\boldsymbol{a}$-labeling** if for every $\hat{P}^{(j-1)}\in\hat{\mathscr{P}}^{(j-1)}$ we have*

$$\varphi^{(j)}\Big(\Big\{P^{(j)}\in\mathscr{P}^{(j)}\colon\ P^{(j)}\subseteq\mathcal{K}_j(\hat{P}^{(j-1)})\Big\}\Big)=[a_j]\,.$$

*Then $\hat{\boldsymbol{x}}_K=(\boldsymbol{x}_K^{(1)},\dots,\boldsymbol{x}_K^{(k-1)})\in\hat{A}(k-1,\boldsymbol{a})$ defined in (4) and (5) defines an equivalence relation on $\mathrm{Cross}_k(\mathscr{P}^{(1)})$ (see (7)).*

*Consequently, such a labeling $\boldsymbol{\varphi}$ defines a bijection between $\hat{A}_{\ne\emptyset}$ and $\hat{\mathscr{P}}^{(k-1)}_{\ne\emptyset}$ (see paragraph below (8)) which can be extended to a bijection between $\hat{A}(k-1,\boldsymbol{a})$ and $\hat{\mathscr{P}}^{(k-1)}$ such that*

- *(a) $\hat{\boldsymbol{x}}\in\hat{A}(k-1,\boldsymbol{a})\mapsto\hat{P}^{(k-1)}(\hat{\boldsymbol{x}})\in\hat{\mathscr{P}}^{(k-1)}$ and*
- *(b) if $\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\boldsymbol{x}}))\ne\emptyset$, then $\boldsymbol{P}(\hat{\boldsymbol{x}})=\boldsymbol{P}(K)$ for some $K\in\hat{P}^{(k-1)}(\hat{\boldsymbol{x}})$ is well defined,*
- *(c) $K\in\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\boldsymbol{x}}_K))$ for every $K\in\mathrm{Cross}_k(\mathscr{P}^{(1)})$, and*
- *(d) $\boldsymbol{P}(\hat{\boldsymbol{x}})=\{P^{(j)}\}_{j=1}^{k-1}$ is a $(k,k-1)$-complex with $P^{(k-1)}=\hat{P}^{(k-1)}(\hat{\boldsymbol{x}})$.*

3.1.2. *Similarity of hypergraphs.* The following definition will enable us to compare hypergraphs of different sizes. Roughly speaking, we will think of two hypergraphs of being "similar" if there exists an integer vector $\boldsymbol{a}$ so that for each of them there exists an $\boldsymbol{a}$-labeled family of partitions on there respective vertex sets such that for every $\hat{\boldsymbol{x}} \in \hat{A}(k-1,\boldsymbol{a})$ the hypergraphs have the similar density on the respective polyad with address $\hat{\boldsymbol{x}}$.

**Definition 22** (**$(d_{\boldsymbol{a},k},\varepsilon)$-partition**)**.** *Suppose $\varepsilon>0$, $\boldsymbol{a}=(a_1,\dots,a_{k-1})$ is a vector of positive integers, $\hat{A}(k-1,\boldsymbol{a})$ is an address space, $d_{\boldsymbol{a},k}\colon \hat{A}(k-1,\boldsymbol{a})\to[0,1]$ is a* ***density function****, and $H^{(k)}$ is a $k$-uniform hypergraph.*

*We say an $\boldsymbol{a}$-labeled family of partitions $\mathscr{P}=\mathscr{P}(k-1,\boldsymbol{a})$ on $V(H^{(k)})$ is a $(d_{\boldsymbol{a},k},\varepsilon)$**-partition of** $H^{(k)}$ if for every $\hat{\boldsymbol{x}}\in\hat{A}(k-1,\boldsymbol{a})$*

$$d\big(H^{(k)}\big|\hat{P}^{(k-1)}(\hat{\boldsymbol{x}})\big)=d_{\boldsymbol{a},k}(\hat{\boldsymbol{x}})\pm\varepsilon\,.$$

The concepts above allow to define an object similar to the *cluster graph* in the context of Szemerédi's regularity lemma. For a given $\delta>0$ Szemerédi's regularity lemma provides a partition of the vertex set $V=V_1\cup\dots\cup V_t$ of a given graph $G$, so that all but $\delta t^2$ pairs $(V_i,V_j)$ are $(\delta,*,1)$-regular. For many applications of that lemma it suffices to "reduce" the whole graph to a weighted graph on $[t]$, where the weight of the edge $ij$ corresponds to the density of the bipartite subgraph of $G$ induced on $(V_i,V_j)$ (usually it will also be useful to mark those edges which correspond to irregular pairs). With that notion of cluster graph, one may say that two graphs $G_1$ and $G_2$ have the same structure if they admit a regular partition in the same number of parts so that the weights (densities) of the cluster graphs are essentially equal or deviate by at most $\varepsilon$.

The notion of address space extends the concept of the vertex labeling of the cluster graph in the context of the hypergraph regularity lemma. This way the function $d_{\boldsymbol{a},k}$ plays the rôle of the edge weights of the cluster graph. As we considered two graphs to be similar if they admit a regular partition with essentially the same cluster graph, we will view hypergraphs $H_1^{(k)}$ and $H_2^{(k)}$ to be $\varepsilon$-similar if there exists an integer vector $\boldsymbol{a}$ (and hence an address space $\hat{A}(k-1,\boldsymbol{a})$) and a density function function $d_{\boldsymbol{a},k}$ such that there is a "regular" $(d_{\boldsymbol{a},k},\varepsilon)$-partition $\mathscr{P}_1(k-1,\boldsymbol{a})$ of $H_1^{(k)}$ and a "regular" $(d_{\boldsymbol{a},k},\varepsilon)$-partition $\mathscr{P}_2(k-1,\boldsymbol{a})$ of $H_2^{(k)}$.

The following lemma, which is a simple corollary of the regularity lemma for hypergraphs, roughly states, that for any given infinite sequence $(H_i^{(k)})_{i=1}^\infty$ of hypergraphs and partitions, there exists a sub-sequence $(H_{j_i}^{(k)})_{i=1}^\infty$ of "similar" hypergraphs (see (*iv*) of Lemma 23) on a "regular family of partitions" (see (*i*) and (*ii*)), which refine the original partitions (see (*iii*)).

**Lemma 23.** *Let $\boldsymbol{a}=(a_1,\dots,a_{k-1})$ be a vector of positive integers. Suppose $(H_i^{(k)})_{i=1}^\infty$ is a sequence of hypergraphs such that $n_i=|V(H_i^{(k)})|\to\infty$ and for every $i\in\mathbb{N}$ there is a family of partitions $\mathscr{Q}_i=\mathscr{Q}_i(k-1,\boldsymbol{a})$ on $V(H_i^{(k)})$ with an equitable vertex partition, $\mathscr{Q}^{(1)}=\{V_1,\dots,V_{a_1}\}$ satisfying $|V_1|\le\dots\le|V_{a_1}|\le|V_1|+1$. Then the following is true.*

*For all positive constants $\mu$ and $\delta_k$ and functions*

$$\delta_j\colon\ (0,1]^{k-j}\to(0,1]\ \textit{for}\ j=2,\dots k-1\ \textit{and}\ r\colon\ \mathbb{N}\times(0,1]^{k-2}\to\mathbb{N}$$

*there exist an integer vector $\boldsymbol{b}=(b_1,\dots,b_{k-1})$, an address space $\hat{A}(k-1,\boldsymbol{b})$, a density function $d_{\boldsymbol{b},k}\colon\ \hat{A}(k-1,\boldsymbol{b})\to[0,1]$, some $d_0>0$ and a sub-sequence $(H_{j_i}^{(k)})_{i=1}^\infty$ of $(H_i^{(k)})_{i=1}^\infty$ such that for every $i\in\mathbb{N}$ there is a vector $\boldsymbol{d}_{j_i}=(d_{j_i,2},\dots,d_{j_i,k-1})\in[0,1]^{k-2}$ and a $\boldsymbol{b}$-labeled family of partitions $\mathscr{P}_{j_i}=\mathscr{P}_{j_i}(k-1,\boldsymbol{b})$ on $V(H_{j_i}^{(k)})$ such that*

(*i*) $\mathscr{P}_{j_i}$ *is* $(\mu, \boldsymbol{\delta}(\boldsymbol{d}_{j_i}), \boldsymbol{d}_{j_i}, r(b_1, \boldsymbol{d}_{j_i}))$*-equitable,*
(*ii*) $H^{(k)}_{j_i}$ *is* $(\delta_k, *, r(b_1, \boldsymbol{d}_{j_i}))$*-regular w.r.t.* $\mathscr{P}_{j_i}$*,*
(*iii*) $\mathscr{P}_{j_i} \prec \mathscr{Q}_{j_i}$*,*
(*iv*) $\min\{d_{j_i,2}, \dots, d_{j_i,k-1}\} \geq d_0$*, and*
(*v*) $\mathscr{P}_{j_i}$ *is a* $(d_{\boldsymbol{b},k}, \mu)$*-partition of* $H^{(k)}_{j_i}$*.*

Since we introduced three concepts related to the partitions, before we start with the proof, we briefly recall their meaning. Part $(i)$ of Lemma 23 describes the *regularity* properties of the auxiliary partition $\mathscr{P}_{j_i}$ (see Definition 18) and $(ii)$ describes the regularity of the hypergraph $H^{(k)}_{j_i}$ w.r.t. the partition $\mathscr{P}_{j_i}$ (see Definition 19). Finally $(v)$ states that the *densities* of all $H_{j_i}$ $(i \in \mathbb{N})$ on polyads with the same address are essentially the same and described by the function $d_{\boldsymbol{b},k}$ (see Definition 22).

*Proof.* Note that for given input parameters $S = \max_{j\in[k-1]} a_j$, $\mu$, and $\delta_k$ and functions $\delta_j$ and $r$ the regularity lemma, Theorem 20, guarantees for every $i \in \mathbb{N}$ the existence of a family of partitions $\mathscr{P}_i$ on $H^{(k)}_i$ with properties $(i)$–$(iv)$ for some $\boldsymbol{b} = \boldsymbol{b}_i$ (which may depend on $i$) and some $d_0$ (independent of $i$).

The proof of Lemma 23 relies on the observation that it suffices to consider only finitely many choices for the integer vector $\boldsymbol{b}$ and for the density function $d_{\boldsymbol{b},k}$ (in view of $(v)$), which implies that for an infinite sub-sequence of $(H^{(k)}_i)_{i=1}^\infty$ those choices must be the same. It is obvious, that there are only finitely many choices for $\boldsymbol{b}$ as Theorem 20 gives an upper bound $T_0$ on $\max_{j\in[k-1]} b_j$, which is independent of $H^{(k)}_i$. However, the function $d_{\boldsymbol{b},k}$ is real-valued and we have to use an appropriate discretization. In view of Definition 21, one possible discretization is to consider intervals in $[0,1]$ of length about $2\mu$. More precisely, let $\mu_0 \in (0,1]$ such that $\lceil 1/(2\mu) \rceil = 1/(2\mu_0)$ and for every $\boldsymbol{b}$ consider special density functions

$$d_{\boldsymbol{b},k} \colon\ \hat{A}(k-1, \boldsymbol{b}) \to \{(2j-1)\mu_0 \colon\ j = 1, \dots, 1/(2\mu_0)\}\,. \tag{9}$$

Clearly, for every $\boldsymbol{b}$ there are only finitely many such density functions and, on the other hand, for any hypergraph $H^{(k)}_i$ and any $\boldsymbol{b}_i$-labeled family of partitions $\mathscr{P}_i(k-1, \boldsymbol{b}_i)$ there exist at least one such special function $\boldsymbol{d}_{\boldsymbol{b}_i,k}$ so that $(v)$ holds.

Summarizing, since any given $S = \max_{j\in[k-1]} a_j$ and input parameters $\mu$, $\delta_k$ and functions $\delta_j$ and $r$ after an application of Theorem 20 to an $S$-bounded $\mathscr{Q}_i$ and $H^{(k)}_i$ the entries of the resulting $\boldsymbol{b}_i$ is bounded by $T_0$ there exist some particular vector $\boldsymbol{b}$ and an infinite sub-sequence $(H^{(k)}_{j_i})_{i=1}^\infty$ and a sequence of partitions $(\mathscr{P}_{j_i})_{i=1}^\infty$ such that properties $(i)$–$(iv)$ hold. Considering then only density functions $d_{\boldsymbol{b},k}$ as in (9), we infer the existence of some function $d_{\boldsymbol{b},k}$ and the existence of some infinite sub-sequences of $(H^{(k)}_{j_i})_{i=1}^\infty$ and $(\mathscr{P}_{j_i})_{i=1}^\infty$ such that $(v)$ holds. □

3.2. **Index of a partition.** In this section we recall the notion of *index* (or *mean-square density*) of a family of partition, which plays a crucial rôle in the proofs of the aforementioned (hyper)graph regularity lemmas.

**Definition 24** (**index**)**.** *Let* $H^{(k)}$ *be a* $k$*-uniform hypergraph on* $n$ *vertices and* $\mathscr{P}$ *be a family of partitions on* $V(H^{(k)})$*. The* ***index of*** $\mathscr{P}$ ***w.r.t.*** $H^{(k)}$ *is defined by*

$$\operatorname{ind}(\mathscr{P}|H^{(k)}) = \frac{1}{\binom{n}{k}} \sum \left\{ d^2(H^{(k)}|\hat{P}^{(k-1)})\big|\mathcal{K}_k(\hat{P}^{(k-1)})\big| \colon\ \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \right\}.$$

As an immediate consequence from the definition of index we have

$$0 \leq \operatorname{ind}(\mathscr{P}|H^{(k)}) \leq 1 \tag{10}$$

for every hypergraph $H^{(k)}$ and every family of partitions $\mathscr{P}$ on $V(H^{(k)})$. The following is a simple consequence of the Cauchy–Schwarz inequality.

**Fact 25.** *If $H^{(k)}$ is a $k$-uniform hypergraph and $\mathscr{P} \prec \mathscr{Q}$ are two refining families of partitions on $V(H^{(k)})$, then $\operatorname{ind}(\mathscr{P}|H^{(k)}) \geq \operatorname{ind}(\mathscr{Q}|H^{(k)})$.*

A proof of Fact 25 can be found in [25, Lemma 10.3].

The main lemma of this section, Lemma 27, considers two refining partitions, with "almost" the same index. For the statement of that lemma we need the following definition.

**Definition 26** ($\nu$**-misbehaved**)**.** *Let $\nu > 0$ and $\mathscr{P} \prec \mathscr{Q}$ be two refining families of partitions on the same vertex set. We say a polyad $\hat{Q}^{(k-1)} \in \hat{\mathscr{Q}}^{(k-1)}$ is $\nu$-**misbehaved** w.r.t. $\mathscr{P}$, if*

$$\sum \Big\{ \big|\mathcal{K}_k(\hat{P}^{(k-1)})\big| \colon \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}\,,\ \hat{P}^{(k-1)} \subseteq \hat{Q}^{(k-1)}\,, \qquad \text{and } \big|d(H^{(k)}|\hat{P}^{(k-1)}) - d(H^{(k)}|\hat{Q}^{(k-1)})\big| > \nu \Big\} \geq \nu \big|\mathcal{K}_k(\hat{Q}^{(k-1)})\big|\,. \tag{11}$$

*We denote by $\mathrm{MB}_{\mathscr{P}}(\mathscr{Q},\nu)$ the set of all $\nu$-misbehaved polyads $\hat{Q}^{(k-1)} \in \hat{\mathscr{Q}}^{(k-1)}$.*

The following is the main lemma of the section. It asserts that if the index of two refining partitions is "close," then there are only few misbehaved polyads in the coarser partition.

**Lemma 27.** *Let $\varepsilon$, $\nu > 0$, $H^{(k)}$ be a $k$-uniform hypergraph on $n$ vertices and $\mathscr{P} \prec \mathscr{Q}$ be two refining families of partitions on $V(H^{(k)})$. If $\operatorname{ind}(\mathscr{P}|H^{(k)}) \leq \operatorname{ind}(\mathscr{Q}|H^{(k)}) + \varepsilon$, then*

$$\sum \Big\{ \big|\mathcal{K}_k(\hat{Q}^{(k-1)})\big| \colon \hat{Q}^{(k-1)} \in \mathrm{MB}_{\mathscr{P}}(\mathscr{Q},\nu) \Big\} \leq \frac{2\varepsilon}{\nu^3}\binom{n}{k}\,.$$

The proof of Lemma 27 relies on the following well known lemma, which is the defect form of the Cauchy–Schwarz inequality.

**Lemma 28.** *Suppose $\emptyset \neq J \subseteq I$ are index sets. For every $i \in I$ let $\sigma_i$ and $d_i$ be arbitrary non-negative reals and let $\sigma_I = \sum_{i\in I}\sigma_i$ and $\sigma_J = \sum_{j\in J}\sigma_j$. If for some (not necessarily positive) $\alpha \in \mathbb{R}$*

$$\sum_{j\in J}\frac{\sigma_j}{\sigma_J}d_j = \sum_{i\in I}\frac{\sigma_i}{\sigma_I}d_i + \alpha\,, \quad \textit{then} \quad \sum_{i\in I}\sigma_i d_i^2 \geq \sigma_I\Big(\sum_{i\in I}\frac{\sigma_i}{\sigma_I}d_i\Big)^2 + \alpha^2\sigma_J\,.$$

*Proof of Lemma 27.* Let $\hat{Q}^{(k-1)} \in \mathrm{MB}_{\mathscr{P}}(\mathscr{Q},\nu)$ be fixed and let the hypergraphs $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ with $\hat{P}^{(k-1)} \subseteq \hat{Q}^{(k-1)}$ be indexed by some set $I$ and set for every $i \in I$

$$d_i = d(H^{(k)}|\hat{P}_i^{(k-1)}) \quad \text{and} \quad \sigma_i = |\mathcal{K}_k(\hat{P}_i^{(k-1)})|\,.$$

Clearly, with $\sigma_I = |\mathcal{K}_k(\hat{Q}^{(k-1)})|$ and $d = d(H^{(k)}|\hat{Q}^{(k-1)})$ we have $\sum_{i\in I}\sigma_i = \sigma_I$ and

$$|H^{(k)} \cap \mathcal{K}_k(\hat{Q}^{(k-1)})| = d\sigma_I = \sum_{i\in I} d_i\sigma_i\,. \tag{12}$$

Moreover, (11) corresponds to $\sum\{\sigma_j \colon |d_j - d| > \nu\} \geq \nu\sigma_I$ and, consequently, for some $J \subseteq I$ we obtain

$$\Big|\sum_{j\in J}\frac{\sigma_j}{\sigma_J}d_j - \sum_{i\in I}\frac{\sigma_i}{\sigma_I}d_i\Big| = \Big|\sum_{j\in J}\frac{\sigma_j}{\sigma_J}d_j - d\Big| \geq \nu\,.$$

where $\sigma_J$ is defined as $\sigma_J = \sum_{j\in J}\sigma_j$ and $J$ satisfies

$$\sigma_J \geq \frac{\nu}{2}\sigma_I$$

Therefore, Lemma 28 implies

$$\sum_{i\in I} \sigma_i d_i^2 \geq \sigma_I \left( \sum_{i\in I} \frac{\sigma_i}{\sigma_I} d_i \right)^2 + \frac{\nu^3}{2}\sigma_I \,. \tag{13}$$

Summarizing, due to (13) and (12) we just showed for each $\hat{Q}^{(k-1)} \in \mathrm{MB}_{\mathscr{P}}(\mathscr{Q},\nu)$ that

$$\sum \left\{ d^2(H^{(k)}|\hat{P}^{(k-1)})|\mathcal{K}_k(\hat{P}^{(k-1)})| \colon \; \hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \text{ and } \hat{P}^{(k-1)} \subseteq \hat{Q}^{(k-1)} \right\}$$
$$\geq d^2(H^{(k)}|\hat{Q}^{(k-1)})|\mathcal{K}_k(\hat{Q}^{(k-1)})| + \frac{\nu^3}{2}|\mathcal{K}_k(\hat{Q}^{(k-1)})| \,.$$

Consequently, we infer from Lemma 28 (applied to every $\hat{Q}^{(k-1)} \notin \mathrm{MB}_{\mathscr{P}}(\mathscr{Q},\nu)$ with $\alpha = 0$) and the last inequality (applied to every $\hat{Q}^{(k-1)} \in \mathrm{MB}_{\mathscr{P}}(\mathscr{Q},\nu)$) that

$$\mathrm{ind}(\mathscr{P}|H^{(k)}) \geq \mathrm{ind}(\mathscr{Q}|H^{(k)}) + \frac{\nu^3}{2\binom{n}{k}} \sum \left\{ \left|\mathcal{K}_k(\hat{Q}^{(k-1)})\right| \colon \; \hat{Q}^{(k-1)} \in \mathrm{MB}_{\mathscr{P}}(\mathscr{Q},\nu) \right\}$$

and hence the assumption of Lemma 27 implies

$$\sum \left\{ \left|\mathcal{K}_k(\hat{Q}^{(k-1)})\right| \colon \; \hat{Q}^{(k-1)} \in \mathrm{MB}_{\mathscr{P}}(\mathscr{Q},\nu) \right\} \leq \frac{2\varepsilon}{\nu^3}\binom{n}{k} \,.$$

□

## 4. Proof of the main result

4.1. **Proof of Theorem 6.** In our argument we will assume that Theorem 6 fails. This means that there exists a family of $k$-uniform hypergraphs $\mathscr{F}$ and a constant $\eta > 0$ such that for every $c$, $C$, and $n_0$ there exists a hypergraph $H^{(k)}$ on $n \geq n_0$ vertices which is $\eta$-far from $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ and which for every $\ell \leq C$ contains at most $cn^\ell$ induced copies of $F^{(k)}$ for every $F^{(k)} \in \mathscr{F}$ on $\ell$ vertices. Applying this assumption successively with $c = 1/i$ and $C = i$ for $i \in \mathbb{N}$ yields the following fact.

**Fact 29.** *If Theorem 6 fails for a family of $k$-uniform hypergraphs $\mathscr{F}$ and $\eta > 0$, then there exists a sequence of hypergraphs $(H_i^{(k)})_{i=1}^\infty$ with $n_i = |V(H_i^{(k)})| \to \infty$ such that for every $i \in \mathbb{N}$*

- *(i) $H_i^{(k)}$ is $\eta$-far from $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ and*
- *(ii) for every $\ell \leq i$ and every $F^{(k)} \in \mathscr{F}$ with $|V(F^{(k)})| = \ell$ the number of induced copies of $F^{(k)}$ in $H_i^{(k)}$ is less than $n_i^\ell/i$.*

The same assumption (for graphs) was considered by Lovász and B. Szegedy [19]. While they derived a contradiction based on the properties of a "limit object" of a sub-sequence of $(H_i^{(k)})_{i=1}^\infty$ the existence of which was established in [18], here we will only consider hypergraphs of the sequence $(H_i^{(k)})_{i=1}^\infty$. More precisely, the following, main lemma in the proof of Theorem 6, will locate two special hypergraphs $I^{(k)} = H_i^{(k)}$ and $J^{(k)} = H_j^{(k)}$ in the sequence from which we derive a contradiction.

**Lemma 30.** *Suppose Theorem 6 fails for $\mathscr{F}$ and $\eta > 0$. Then there exists a hypergraph $I = I^{(k)}$ on $\ell$ vertices, an integer vector $\boldsymbol{a} = (a_1, \ldots, a_{k-1})$, a density function $d_{\boldsymbol{a},k} \colon \hat{A}(k-1,\boldsymbol{a}) \to [0,1]$, and a family of partitions $\mathscr{Q}_I = \mathscr{Q}_I(k-1,\boldsymbol{a})$ on $V(I^{(k)})$ such that*

- *(I1) $\mathscr{Q}_I$ is a $(d_{\boldsymbol{a},k}, \eta/24)$-partition of $I^{(k)}$,*
- *(I2) $|\mathrm{Cross}_k(\mathscr{Q}_I^{(1)})| \geq (1 - \frac{\eta}{24})\binom{\ell}{k}$, and*
- *(I3) $I^{(k)}$ is $\eta$-far from $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$.*

*Furthermore, there exists a hypergraph* $J = J^{(k)}$ *on* $n \geq \ell$ *vertices, a family of partitions* $\mathscr{Q}_J = \mathscr{Q}_J(k-1, \boldsymbol{a})$ *on* $V(J^{(k)})$*, an integer vector* $\boldsymbol{b} = (b_1, \dots, b_{k-1})$*, and a family of partitions* $\mathscr{P}_J = \mathscr{P}_J(k-1, \boldsymbol{b})$ *on* $V(J^{(k)})$ *such that*

(*J1*) $\mathscr{Q}_J$ *is a* $(d_{\boldsymbol{a},k}, \eta/24)$*-partition of* $J^{(k)}$ *and*
(*J2*) $\mathscr{P}_J \prec \mathscr{Q}_J$.

*Moreover, there exists an* $\ell$*-set* $L \in \mathrm{Cross}_\ell(\mathscr{P}_J^{(1)})$ *such that*

(*L1*) $|L \cap V_i| = |U_i|$ *where* $\mathscr{Q}_I^{(1)} = \{U_i \colon i \in [a_1]\}$ *and* $\mathscr{Q}_J^{(1)} = \{V_i \colon i \in [a_1]\}$,
(*L2*)

$$\Big|\big\{K \in \tbinom{L}{k} \cap \mathrm{Cross}_k(\mathscr{Q}_J^{(1)}) \colon \\ |d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) - d(J^{(k)}|\hat{Q}_J^{(k-1)}(K))| > \tfrac{\eta}{12}\big\}\Big| \leq \tfrac{4\eta}{9}\tbinom{\ell}{k},$$

(*L3*) *any* $k$*-uniform hypergraph* $G^{(k)}$ *with vertex set* $L$ *and with the property*

$$K \in G^{(k)} \Rightarrow d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) \geq \tfrac{\eta}{12}$$
$$\textit{and}\quad K \notin G^{(k)} \Rightarrow d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) \leq 1 - \tfrac{\eta}{12},$$

*belongs to* $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$.

For the proof of Lemma 30 we will successively chose sub-sequences of $(H_i^{(k)})_{i=1}^\infty$ (see Fact 29), with each sequence being a sub-sequence of the previous. The sub-sequences will be obtained by Lemma 23 and after finitely many iterations we will select $I^{(k)}$ and $J^{(k)}$ from the "most current" sub-sequence (from which properties (*I1*-*I3*) and (*J1*-*J2*) will follow). We stop the iterations when the last sub-sequence $(H_{j_i}^{(k)})_{i=1}^\infty$ satisfies for every $i \in \mathbb{N}$

$$\mathrm{ind}(\mathscr{P}_{j_i}|H_{j_i}^{(k)}) \leq \mathrm{ind}(\mathscr{Q}_{j_i}|H_{j_i}^{(k)}) + \varepsilon \tag{14}$$

for some appropriately chosen $\varepsilon = \varepsilon(\eta)$. Clearly, we will reach this situation after at most $1/\varepsilon$ iterations (see (10) and Fact 25). By Lemma 27, we will infer from (14) that a randomly selected $\ell$-tuple from the set of all $\ell$-tuples satisfying (*L1*) admits (*L2*). Moreover, if we select $J^{(k)}$ "far enough" in the sequence, then (*ii*) of Fact 29 will be the key for proving (*L3*). We give the precise details in Section 4.2 and below we derive the main result of this paper from Lemma 30.

*Proof of Theorem 6.* The proof is by contradiction. Suppose there exists a family of $k$-uniform hypergraphs $\mathscr{F}$ and some $\eta > 0$ so that Theorem 6 fails. We apply Lemma 30 which yields hypergraphs $I^{(k)}$ (on $\ell$ vertices) and $J^{(k)}$ (on $n$ vertices) and an $\ell$-set $L \subseteq V(J^{(k)})$. In view of property (*L3*) we will define a hypergraph $G^{(k)}$ on the vertex set $L$. In order to obtain the desired contradiction we will "compare" the $\ell$-vertex hypergraph $G^{(k)}$ with the $\ell$-vertex hypergraph $I^{(k)}$. For that we need some bijection $\psi$ from $L$ to $V(I^{(k)})$. We will chose some bijection $\psi$ which "agrees" with the labellings of $\mathscr{Q}_J$ and $\mathscr{Q}_I$, i.e., we require that for any $k$-tuple $K \in \mathrm{Cross}_k(\mathscr{Q}_J^{(1)})$ the address $\hat{\boldsymbol{x}}_K$ (see Definition 21) of $K$ w.r.t. the $\boldsymbol{a}$-labeled partition $\mathscr{Q}_J$ coincides with the address $\hat{\boldsymbol{x}}_{\psi(K)}$ of $\psi(K)$ w.r.t. the $\boldsymbol{a}$-labeled partition $\mathscr{Q}_I$. More precisely, fix a bijection $\psi \colon L \to V(I^{(k)})$ such that for every $K \in \binom{L}{k}$ the following holds: if $K \in \mathrm{Cross}_k(\mathscr{Q}_J^{(1)})$ then

$$\psi(K) \in \mathrm{Cross}_k(\mathscr{Q}_I^{(1)}) \quad \text{and} \quad \hat{\boldsymbol{x}}_K = \hat{\boldsymbol{x}}_{\psi(K)}. \tag{15}$$

For a subset of $E \subseteq \binom{L}{k}$ we set $\psi(E) = \{\psi(K) \colon K \in E\}$.

We then define the hypergraph $G^{(k)}$ on $L$ by

$$K \in G^{(k)} \quad \Longleftrightarrow \quad \begin{cases} \text{either} & d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) \geq \frac{\eta}{12} \text{ and } \psi(K) \in I^{(k)} \\ \text{or} & d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) > 1 - \frac{\eta}{12}\,. \end{cases} \tag{16}$$

for every $k$-tuple $K \in \binom{L}{k}$. Consequently, by *(L3)* of Lemma 30

$$G^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})\,. \tag{17}$$

It is left to show

$$\left|I^{(k)} \triangle \psi(G^{(k)})\right| \leq \eta \binom{\ell}{k}, \tag{18}$$

which due to (17) contradicts *(I3)* of Lemma 30, i.e., (18) contradicts the fact that $I^{(k)}$ is $\eta$-far from $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$.

We cover the symmetric difference $I^{(k)} \triangle \psi(G^{(k)})$ by four sets $D_1, \dots, D_4$ defined by

$$\begin{aligned} D_1 &= \binom{V(I^{(k)})}{k} \setminus \mathrm{Cross}_k(\mathscr{Q}_I^{(1)})\,, \\ D_2 &= \psi\Big(\{K \in \tbinom{L}{k} \cap \mathrm{Cross}_k(\mathscr{Q}_J^{(1)}) \colon \\ &\qquad\qquad |d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) - d(J^{(k)}|\hat{Q}_J^{(k-1)}(K))| > \eta/12\}\Big)\,, \\ D_3 &= I^{(k)} \cap \bigcup \Big\{\mathcal{K}_k(\hat{Q}_I^{(k-1)}) \colon\ d(I^{(k)}|\hat{Q}_I^{(k-1)}) < \eta/4\Big\}\,, \end{aligned}$$

and

$$D_4 = \binom{L}{k} \setminus \Big(I^{(k)} \cap \bigcup \Big\{\mathcal{K}_k(\hat{Q}_I^{(k-1)}) \colon\ d(I^{(k)}|\hat{Q}_I^{(k-1)}) > 1 - \eta/4\Big\}\Big)\,.$$

We first show that indeed $I^{(k)} \triangle \psi(G^{(k)}) \subseteq D_1 \cup \dots \cup D_4$. For that first consider some $K' \in I^{(k)} \setminus \psi(G^{(k)})$ and set $K = \psi^{-1}(K')$. By the definition of $G^{(k)}$ in (16) we have $d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) < \frac{\eta}{12}$. Then it is easy to show that if $K' \notin D_1 \cup D_2$ then $K' \in D_3$. Indeed, we have:

$$\begin{aligned} &K' \in I^{(k)} \setminus \big(\psi(G^{(k)}) \cup D_1 \cup D_2\big) \\ &\qquad\overset{(16)}{\Longrightarrow} d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) < \frac{\eta}{12} \overset{K' \notin D_1 \cup D_2}{\Longrightarrow} d(J^{(k)}|\hat{Q}_J^{(k-1)}(K)) < \frac{\eta}{6}\,. \end{aligned} \tag{19}$$

Due to *(J1)* and *(I1)* of Lemma 30, both $\mathscr{Q}_J$ and $\mathscr{Q}_I$ are $(d_{\boldsymbol{a},k}, \eta/24)$-partitions with the same $\hat{A}(k-1, \boldsymbol{a})$ and the same density function $d_{\boldsymbol{a},k} \colon \hat{A}(k-1,\boldsymbol{a}) \to [0,1]$.

Hence, on the one hand, we infer $d(J^{(k)}|\hat{Q}_J^{(k-1)}(K)) = d_{\boldsymbol{a},k}(\hat{\boldsymbol{x}}_K) \pm \eta/24$ and, on the other hand, due to (15) and $K = \psi^{-1}(K')$, we have $d(I^{(k)}|\hat{Q}_I^{(k-1)}(K')) = d_{\boldsymbol{a},k}(\hat{\boldsymbol{x}}_K) \pm \eta/24$. Thus, $|d(J^{(k)}|\hat{Q}_J^{(k-1)}(K)) - d(I^{(k)}|\hat{Q}_I^{(k-1)}(K'))| \leq \eta/12$ and the right-hand side of (19) implies

$$\begin{aligned} &K' \in I^{(k)} \setminus \big(\psi(G^{(k)}) \cup D_1 \cup D_2\big) \overset{(19)}{\Longrightarrow} d(J^{(k)}|\hat{Q}_J^{(k-1)}(K)) < \frac{\eta}{6} \\ &\qquad\overset{(J1)\&(I1)}{\Longrightarrow} d(I^{(k)}|\hat{Q}_I^{(k-1)}(K')) < \frac{\eta}{6} + \frac{\eta}{12} = \frac{\eta}{4} \implies K' \in D_3\,. \end{aligned}$$

Similarly, for $K' \in \psi(G^{(k)}) \setminus I^{(k)}$ and $K = \psi^{-1}(K')$ we infer by similar arguments as above:

$$\begin{aligned} &K' \in \psi(G^{(k)}) \setminus \big(I^{(k)} \cup D_1 \cup D_2\big) \\ &\overset{(16)}{\Longrightarrow} d(J^{(k)}|\hat{P}_J^{(k-1)}(K)) > 1 - \frac{\eta}{12} \overset{K' \notin D_1 \cup D_2}{\Longrightarrow} d(J^{(k)}|\hat{Q}_J^{(k-1)}(K)) > 1 - \frac{\eta}{6} \\ &\qquad\qquad \overset{(J1)\&(I1)}{\Longrightarrow} d(I^{(k)}|\hat{Q}_I^{(k-1)}(K')) > 1 - \frac{\eta}{4} \Longrightarrow K' \in D_4 \,. \end{aligned}$$

Consequently, $I^{(k)} \triangle \psi(G^{(k)}) \subseteq D_1 \cup \dots \cup D_4$. Moreover, from $(I2)$ of Lemma 30 we infer $|D_1| = |\binom{V(I^{(k)})}{k} \setminus \mathrm{Cross}_k(\mathscr{Q}_I^{(1)})| \le \eta\binom{\ell}{k}/24$ and $(L2)$ implies $|D_2| \le 4\eta\binom{\ell}{k}/9$. Finally, the definitions of $D_3$ and $D_4$ yield $|D_3| \le \eta\binom{\ell}{k}/4$ and $|D_3| \le \eta\binom{\ell}{k}/4$. Summarizing the above, we obtain

$$\big|I^{(k)} \triangle \psi(G^{(k)})\big| \le |D_1| + |D_2| + |D_3| + |D_4| \le \Big(\frac{\eta}{24} + \frac{4\eta}{9} + \frac{\eta}{4} + \frac{\eta}{4}\Big)\binom{\ell}{k} < \eta\binom{\ell}{k} \,.$$

Thus we proved (18), which together with (17) yields a contradiction to $(I3)$ of Lemma 30. □

4.2. **Proof of main lemma.** Since the proof is a bit technical, we will first give a sketch. The proof of Lemma 30 is based on iterative applications of Lemma 23. Given an infinite sequence of hypergraphs $(H_i^{(k)})_{i=1}^{\infty}$ each with a partition $\mathscr{Q}_i(k-1, \boldsymbol{a})$ (where $\boldsymbol{a}$ is the same for every $i \in \mathbb{N}$) and "measures of precision" (constants $\mu$, $\delta_k$ and functions $\delta_{k-1}, \dots, \delta_2$), Lemma 23 guarantees a vector $\boldsymbol{b}$ and a function $d_{\boldsymbol{b},k}\colon \hat{A}(k-1, \boldsymbol{b}) \to [0,1]$, a subsequence $(H_{j_i}^{(k)})_{i=1}^{\infty}$ of $(H_i^{(k)})_{i=1}^{\infty}$ and $\boldsymbol{b}$-labeled partitions $\mathscr{P}_{j_i}(k-1, \boldsymbol{b}) \prec \mathscr{Q}_{j_i}$ such that

- $(a)$ $\mathscr{P}_{j_i}$ is "sufficiently regular" and
- $(b)$ $\mathscr{P}_{j_i}$ is $(d_{\boldsymbol{b},k}, \mu)$-partition of $H_{j_i}^{(k)}$.

We will show that after at most $1/\varepsilon$ iterations, we will get two consecutive partitions $\mathscr{P}_{j_i} \prec \mathscr{Q}_{j_i}$ with the refining polyads having similar densities, more precisely $\mathscr{P}_{j_i}$ and $\mathscr{Q}_{j_i}$ will satisfy the assumptions of Lemma 27. We then set $J^{(k)}$ equal to $H_{j_i}^{(k)}$ (for some appropriately chosen $i$) and $I^{(k)}$ equal to the smallest hypergraph of the last sequence $(H_i^{(k)})_{i=1}^{\infty}$ (to which we applied Lemma 23 in the last application). Then Lemma 27 will imply that a random $\ell$-tuple (chosen uniform at random from all $\ell$-sets satisfying $(L1)$) will exhibit property $(L2)$. Moreover, since by part $(ii)$ of Fact 29, which holds since we assume that Theorem 6 fails, $J^{(k)} = H_{j_i}^{(k)}$ contains only a "few" induced copies of forbidden hypergraphs $F^{(k)} \in \mathscr{F}$ and, hence, the counting lemma (in form of Corollary 15) will yield $(L3)$ of Lemma 30.

*Proof of Lemma 30.* Let $\mathscr{F}$ be a family of $k$-uniform hypergraphs and $\eta > 0$ and suppose Theorem 6 fails for $\mathscr{F}$ and $\eta$. By Fact 29 there exist a sequence of hypergraphs $(H_i^{(k)})_{i=1}^{\infty}$ with $n_i = |V(H_i^{(k)})| \to \infty$ admitting properties $(i)$ and $(ii)$ of Fact 29. Without loss of generality we may assume that

$$|V(H_i^{(k)})|^k = n_i^k \le \tfrac{3}{2} n_i \times \dots \times (n_i - k + 1) \tag{20}$$

for every $i \in \mathbb{N}$. In the proof we need an auxiliary constant $\varepsilon$ defined by

$$\varepsilon = \frac{1}{6}\left(\frac{\eta}{12}\right)^4 . \tag{21}$$

We want to iterate Lemma 23. This lemma locates a sub-sequence $(H_{j_i}^{(k)})_{i=1}^{\infty}$ of hypergraphs satisfying $(i)$–$(v)$ of Lemma 23 within a sequence of hypergraphs $(H_i^{(k)})_{i=1}^{\infty}$. Note that in particular property $(i)$ of the sub-sequence $(H_{j_i}^{(k)})_{i=1}^{\infty}$ yields (among other things) the assumption allowing the next iteration, i.e., after an appropriate renaming and relabeling $(i)$ implies that there exist an integer vector $\boldsymbol{a}$ and for every $i \in \mathbb{N}$ there is a family of partitions $\mathscr{Q}_i = \mathscr{Q}_i(k-1, \boldsymbol{a})$ on $V(H_i^{(k)})$ each of them having an equitable vertex partition (see $(b)$ of Definition 18).

For the first iteration we simply choose $\boldsymbol{a} = (1, \dots, 1) \in \mathbb{N}^{k-1}$ and for every $i \in \mathbb{N}$ we let $\mathscr{Q}_i = \mathscr{Q}_i(k-1, \boldsymbol{a})$ be the trivial partition $\mathscr{Q}_i = \{\{V(H_i^{(k)}\}, \{\emptyset\}, \dots, \{\emptyset\}\}$ on $V(H_i^{(k)})$ with only one vertex class and $\mathrm{Cross}_j(\mathscr{Q}_i^{(1)})$ being empty for $j \geq 2$.

More generally, suppose that after $p-1 \geq 0$ iterations we are given an integer vector $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ and sequences $(H_i^{(k)})_{i=1}^{\infty}$ and $(\mathscr{Q}_i)_{i=1}^{\infty}$ such that $\mathscr{Q}_i = \mathscr{Q}_i(k-1, \boldsymbol{a})$ is a family of partitions on $V(H_i^{(k)})$. We will now choose $\mu$, $\delta_k$, and functions $\delta_j$ $(j = 2, \dots, k-1)$ and $r$ with which we want to apply Lemma 23 in the $p$-th iteration. For that set

$$\ell_p = |V(H_1^{(k)})|\,, \qquad \mu = \min\left\{\tfrac{\eta}{24}, \tfrac{\ell^k}{9k!}\right\}, \tag{22}$$

$$\text{and} \qquad \delta_k = \min\left\{\tfrac{\ell^k}{9k!}, \delta_k(\mathrm{ICL}(\ell_p, \gamma = 1/2, d_k = \eta/12))\right\}, \tag{23}$$

where $\delta_k(\mathrm{ICL}(\ell_p, \gamma = 1/2, d_k = \eta/12))$ is given by the "induced counting lemma," Corollary 15, applied for hypergraphs on $\ell_p$ vertices with $\gamma = 1/2$ and $d_k = \eta/12$. Similarly, for $j = 2, \dots, k-1$ let $\delta_j \colon (0,1]^{k-j} \to (0,1]$ be the function in variables $D_j, \dots, D_{k-1}$ given by Corollary 15 for $\ell_p$, $\gamma = 1/2$, and $d_k = \eta/12$, i.e., for $j = 2, \dots, k-1$ we set

$$\delta_j(D_j, \dots, D_{k-1}) = \delta_j(\mathrm{ICL}(\ell_p, \gamma = 1/2, d_k = \eta/12, D_{k-1}, \dots, D_j))\,, \tag{24}$$

$$\text{and} \quad r(B_1, D_2, \dots, D_{k-1}) = r(\mathrm{ICL}(\ell_p, \gamma = 1/2, d_k = \eta/12, D_{k-1}, \dots, D_2))\,, \tag{25}$$

where we make no use of the variable $B_1$ in the definition of $r$. For those choices Lemma 23 yields an integer vector $\boldsymbol{b}$, a density function $d_{\boldsymbol{b},k} \colon \hat{A}(k-1, \boldsymbol{b}) \to [0,1]$, a constant $d_0 > 0$, a sub-sequence $(H_{j_i}^{(k)})_{i=1}^{\infty}$ of $(H_i^{(k)})_{i=1}^{\infty}$, and for every $i \in \mathbb{N}$ a $\boldsymbol{b}$-labeled family partitions $\mathscr{P}_{j_i} = \mathscr{P}_{j_i}(k-1, \boldsymbol{b})$ on $H_{j_i}^{(k)}$ satisfying $(i)$–$(v)$ of Lemma 23. We consider the index (see Definition 24) of the partitions $\mathscr{P}_{j_i}$ and define

$$S_p = \left\{ i \in \mathbb{N} \colon\ \mathrm{ind}(\mathscr{P}_{j_i} | H_{j_i}^{(k)}) \leq \mathrm{ind}(\mathscr{Q}_{j_i} | H_{j_i}^{(k)}) + \varepsilon \right\},$$

where $\varepsilon$ was defined in (21). We distinguish two cases.

If $S_p$ is finite then we iterate Lemma 23 and apply it in the next iteration (after an appropriate relabeling) to the infinite sub-sequence

$$(H_{j_i}^{(k)})_{i \in \mathbb{N} \setminus S_p} \quad \text{with} \quad \ell_{p+1} = |V(H^{(k)}_{\min \mathbb{N} \setminus S_p})|\,.$$

If, on the other hand, $S_p$ is infinite, then we stop iterating. Note that in each iteration the index of $\mathscr{P}_{j_i}$ compared to the index of $\mathscr{Q}_{j_i}$ with respect to $H_{j_i}^{(k)}$ increases by a fixed $\varepsilon$ (chosen independent of $p$) for every $i \in \mathbb{N} \setminus S_p$. Hence, in view of (10), after at most $1/\varepsilon$ iterations the above procedure ends with an infinite set $S_p$.

Let $\boldsymbol{a}$, $(H_i^{(k)})_{i=1}^{\infty}$, $(\mathscr{Q}_i)_{i=1}^{\infty}$, $\boldsymbol{b}$, $d_{\boldsymbol{b},k} \colon \hat{A}(k-1, \boldsymbol{b}) \to [0,1]$, $d_0 > 0$, $(H_{j_i}^{(k)})_{i=1}^{\infty}$, $(\mathscr{P}_{j_i})_{i=1}^{\infty}$, and $S_{p_0}$ be the the input and outcome of that "final," say $p_0$-the, iteration of Lemma 23. In other words for every $i \in S_{p_0}$ we have a $\boldsymbol{b}$-labeled family partitions $\mathscr{P}_{j_i} = \mathscr{P}_{j_i}(k-1, \boldsymbol{b})$ on $H_{j_i}^{(k)}$ satisfying

(L23.$i$) $\mathscr{P}_{j_i}$ is $(\mu, \boldsymbol{\delta}(\boldsymbol{d}_{j_i}), \boldsymbol{d}_{j_i}, r(b_1, \boldsymbol{d}_{j_i}))$-equitable,
(L23.$ii$) $H_{j_i}^{(k)}$ is $(\delta_k, *, r(b_1, \boldsymbol{d}_{j_i}))$-regular w.r.t. $\mathscr{P}_{j_i}$,
(L23.$iii$) $\mathscr{P}_{j_i} \prec \mathscr{Q}_{j_i}$,
(L23.$iv$) $\min\{d_{j_i,2}, \dots, d_{j_i,k-1}\} \geq d_0$, and
(L23.$v$) $\mathscr{P}_{j_i}$ is a $(d_{\boldsymbol{b},k}, \mu)$-partition of $H_{j_i}^{(k)}$,

where $\mu$, $\delta_k$, and functions $\delta_j$ and $r$ were chosen in (22), (23), (24) and (25) depending on $\ell_{p_0} = \ell = |V(H_1^{(k)})|$. Moreover, by the definition of $S_{p_0}$ for every $i \in S_{p_0}$ we have

$$\operatorname{ind}(\mathscr{P}_{j_i}|H_{j_i}^{(k)}) \le \operatorname{ind}(\mathscr{Q}_{j_i}|H_{j_i}^{(k)}) + \varepsilon \tag{26}$$

Without loss of generality we may assume that $p_0 \ge 1$ and consequently due to the choice of $\mu_{p_0-1}$ in (22) and from properties $(i)$ and $(v)$ of the penultimate iteration of Lemma 23 there exist a density function $d_{\boldsymbol{a},k}\colon\ \hat{A}(k-1,\boldsymbol{a}) \to [0,1]$ such that for every $i \in \mathbb{N}$

$$|\operatorname{Cross}_k(\mathscr{Q}_i^{(1)})| \ge \Big(1 - \frac{\eta}{24}\Big)\binom{|V(H_i^{(k)})|}{k} \tag{27}$$

$$\mathscr{Q}_i^{(1)} \text{ is an equitable vertex partition (see } (b) \text{ of Definition 18)} \tag{28}$$

$$\text{and } \mathscr{Q}_i \text{ is a } (d_{\boldsymbol{a},k}, \eta/24)\text{-partition of } H_i^{(k)}\,. \tag{29}$$

Next we choose the special hypergraphs $I^{(k)}$ and $J^{(k)}$ and verify properties (*I1-I3*) and (*J1-J2*). Then we will focus on (*L1-L3*). We set $I^{(k)}$ equal to the first hypergraph in the given sequence for the last iteration, i.e.,

$$I^{(k)} = H_1^{(k)}\,, \quad \ell = \ell_{p_0} = |V(I^{(k)})|\,, \text{ and } \mathscr{Q}_I = \mathscr{Q}_I(k-1,\boldsymbol{a}) = \mathscr{Q}_1(k-1,\boldsymbol{a})\,. \tag{30}$$

Note, however, that due to the relabeling in every iteration $H_1^{(k)}$ in (30) can be different from the first hypergraph in the sequence $(H_i^{(k)})_{i=1}^{\infty}$ originally obtained by Fact 29, which holds since by assumption of Lemma 30 Theorem 6 fails.

Next we select $J^{(k)}$ from the last sub-sequence $(H_{j_i}^{(k)})_{i=1}^{\infty}$. It will be essential for our proof that the selected $J^{(k)}$ contains only a "few" induced copies of forbidden hypergraphs $F^{(k)} \in \mathscr{F}$ on $\ell$ or less vertices. For that we define the auxiliary constant

$$\alpha = \frac{1}{2}\Big(\frac{\eta}{12}\Big)^{\binom{\ell}{k}} \prod_{h=2}^{k-1} d_0^{\binom{\ell}{h}} \times \Big(\frac{1}{b_1}\Big)^{\ell}\,, \tag{31}$$

where $d_0$ is given by Lemma 23 (see (L23.*iv*)). We consider the subset $S_{p_0}^* \subseteq S_{p_0}$ with the property that for every $i \in S_{p_0}^*$

$$\#\Big\{F^{(k)} \overset{\text{ind}}{\subseteq} H_{j_i}^{(k)}\Big\} < \alpha|V(H_{j_i}^{(k)})|^{|V(F^{(k)})|} \text{ for all } F^{(k)} \in \mathscr{F} \text{ with } |V(F^{(k)})| \le \ell\,. \tag{32}$$

In fact $S_{p_0}^*$ is an infinite subset of $S_{p_0}$, since $S_{p_0}$ is infinite and since $(H_{j_i}^{(k)})_{i=1}^{\infty}$ is a sub-sequence of the original sequence of hypergraphs, which satisfy in particular $(ii)$ of Fact 29. For technical reasons we also want the hypergraph $J^{(k)}$ to be large and we select $i_0$ in $S_{p_0}^*$ sufficiently large, so that

$$\frac{1}{b_1}\big|V(H_{j_{i_0}}^{(k)})\big| \ge m_0(\mathrm{ICL}(\ell, \gamma = 1/2, d_k = \eta/12, d_{k-1} = d_0, \dots, d_2 = d_0))\,, \tag{33}$$

where $m_0(\mathrm{ICL}(\ell, \gamma = 1/2, d_k = \eta/12, d_{k-1} = d_0, \dots, d_2 = d_0))$ is given by Corollary 15. We then set

$$J^{(k)} = H_{j_{i_0}}^{(k)}\,,\ n = |V(J^{(k)})|\,,\ \boldsymbol{d}_J = (d_{J,2}, \dots, d_{J,k-1}) = (d_{j_{i_0},2}, \dots, d_{j_{i_0},k-1})\,,$$
$$\mathscr{Q}_J = \mathscr{Q}_J(k-1,\boldsymbol{a}) = \mathscr{Q}_{j_{i_0}}(k-1,\boldsymbol{a}) \text{ and } \mathscr{P}_J = \mathscr{P}_J(k-1,\boldsymbol{b}) = \mathscr{P}_{j_{i_0}}(k-1,\boldsymbol{b})\,.$$

Properties (*I1-I3*) and (*J1-J2*) of Lemma 30 are immediate for those choices of $I^{(k)}$ and $J^{(k)}$. Indeed (*I1*) and (*J1*) follow from (29) and (*I2*) is satisfied due to (27). Property (*I3*) follows from part $(i)$ of Fact 29 and, finally, (*J2*) is a consequence of (L23.*iii*).

It is left to prove the existence of an $\ell$-set $L \in \mathrm{Cross}_\ell(\mathscr{P}_J^{(1)})$ which displays properties (*L1-L3*). For that we consider a random $\ell$-set from $V(J^{(k)})$. More precisely, let the labeled vertex partitions of $\mathscr{Q}_I$ and $\mathscr{Q}_J$ be $\mathscr{Q}_I^{(1)} = \{U_1, \dots, U_{a_1}\}$ and $\mathscr{Q}_J^{(1)} = \{V_1, \dots, V_{a_1}\}$. We select an $\ell$-set uniformly at random from the probability space

$$\Omega = \prod_{i=1}^{a_1} \binom{V_i}{|U_i|},$$

i.e., we select precisely $|U_i|$ vertices from $V_i$ for every $i = 1, \dots, a_1$. Due to that particular choice of $L$, it displays property (*L1*). In view of the other "desired" properties of $L$ we consider the following "bad" events

$B_1\colon\ L \notin \mathrm{Cross}_\ell(\mathscr{P}_J^{(1)})$,

$B_2\colon\ \exists K \in \binom{L}{k} \cap \mathrm{Cross}_k(\mathscr{P}_J^{(1)})\colon\ \boldsymbol{P}_J(K)$ is not a
$(\boldsymbol{\delta}(\boldsymbol{d}_J), \boldsymbol{d}_J, r(b_1, \boldsymbol{d}_J))$-regular $(n/b_1, k, k-1)$-complex,

$B_3\colon\ \exists K \in \binom{L}{k} \cap \mathrm{Cross}_k(\mathscr{P}_J^{(1)})\colon\ J^{(k)}$ is not $(\delta_k, *, r(b_1, \boldsymbol{d}_J))$-regular w.r.t. $\hat{P}_J(K)$,

and

$$B_4\colon\ \Big|\Big\{K \in \tbinom{L}{k} \cap \mathrm{Cross}_k(\mathscr{Q}_J^{(1)})\colon$$
$$|d(J^{(k)}|\hat{P}_J^{(k-1)}(K) - d(J^{(k)}|\hat{Q}_J^{(k-1)}(K))| > \eta/12\Big\}\Big| > \frac{4\eta}{9}\binom{\ell}{k}.$$

Next we estimate the probabilities of the events $B_1, \dots, B_4$. For that the following observation will be useful.

**Fact 31.** *For every $K \in \mathrm{Cross}_k(\mathscr{P}_J^{(1)})$*

$$\mathrm{ext}_L(K) := |\{L \in \Omega\colon\ K \subseteq L\}| = (1 \pm o(1)) \left(\frac{\ell}{n}\right)^k \binom{n/a_1}{\ell/a_1}^{a_1},$$

*where $o(1) \to 0$ as both $\ell$ and $n$ tend to infinity and $a_1$ is fixed.*

*Proof.* Recall that by the definition of $\Omega$, $\mathrm{ext}_L(K)$ is counting for a fixed $k$-set $K$ the number of $\ell$-sets $L$ each of which contain $K$ and for every $i \in [a_1]$ intersect the set $V_i$ in $|U_i| = \ell/a_1$. This number is smallest if $K \subseteq V_i$ for some $i \in [a_1]$ and largest when $|K \cap V_i| \le 1$ for every $i \in [a_1]$. Consequently and (28) we have for every $K \in \binom{V(J^{(k)})}{k}$

$$\binom{n/a_1 - k}{\ell/a_1 - k}\binom{n/a_1}{\ell/a_1}^{a_1-1} \le \mathrm{ext}_L(K) \le \binom{n/a_1 - 1}{\ell/a_1 - 1}^k \binom{n/a_1}{\ell/a_1}^{a_1-k},$$

and straightforward calculations yield Fact 31. □

Without loss of generality we assume that $\ell$ and $n$ are sufficiently large, so that for every $K \in \binom{V(J^{(k)})}{k}$

$$\mathrm{ext}_L(K) = \left(1 \pm \frac{1}{3}\right)\left(\frac{\ell}{n}\right)^k \binom{n/a_1}{\ell/a_1}^{a_1}. \tag{34}$$

(This can easily be achieved by focusing on only sufficiently large hypergraphs in the sub-sequence $(H_{j_i}^{(k)})_{i=1}^\infty$ in the iteration procedure.) We now turn our attention to the events $B_1, \dots, B_4$ and prove upper bounds on the probabilities of those "bad" events. We start with $B_1 \cup B_2 \cup B_3$, i.e., we estimate the events that there is some $k$-set $K \subset L$ such that either $K \notin \mathrm{Cross}_k(\mathscr{P}_J^{(1)})$ or $\boldsymbol{P}_J(K)$ (see (1)) is not regular of $J^{(k)}$ is not regular w.r.t. $\hat{P}_J(K)$.

By (L23.$i$) we have

$$\left|\left\{K\in\tbinom{V(J^{(k)})}{k}\colon\; K\notin \mathrm{Cross}_k(\mathscr{P}_J^{(1)})\right\}\right|\le\mu\binom{n}{k}.$$

Moreover, it follows from (L23.$i$) that

$$\left|\left\{K\in \mathrm{Cross}_k(\mathscr{P}_J^{(1)})\colon\; \boldsymbol{P}_J(K)\text{ is not }(\delta(\boldsymbol{d}_J),\boldsymbol{d}_J,r(b_1,\boldsymbol{d}_J))\text{-regular}\right|\right\}\le\mu\binom{n}{k}$$

and from (L23.$ii$) that

$$\left|\left\{K\in \mathrm{Cross}_k(\mathscr{P}_J^{(1)})\colon\; J^{(k)}\text{ is not }(\delta_k,*,r(b_1,\boldsymbol{d}_J))\text{-regular w.r.t. }\hat{P}_J(K)\right|\right\}\le\delta_k\binom{n}{k}.$$

Due to (34) each $k$-tuple $K\in\binom{V(J^{(k)})}{k}$ extends to at most $(4/3)(\ell/n)^k\binom{n/a_1}{\ell/a_1}^{a_1}$ different $\ell$-sets $L\in\Omega$. Consequently, the number of pairs $(L,K)$, with $L\in\Omega$ and $K\in\binom{L}{k}$ which is *bad*, i.e., $K\notin \mathrm{Cross}_k(\mathscr{P}_J^{(1)})$, or $\boldsymbol{P}_J(K)$ is not regular, or $J^{(k)}$ is not regular w.r.t. $\hat{P}_J(K)$, is at most

$$(2\mu+\delta_k)\binom{n}{k}\times\frac{4}{3}\left(\frac{\ell}{n}\right)^k\binom{n/a_1}{\ell/a_1}^{a_1}. \tag{35}$$

As $|\Omega|=\binom{n/a_1}{\ell/a_1}^{a_1}$, the expected number of bad $k$-tuples $K$ in $\binom{L}{k}$ for a random $\ell$-set $L$ is at most

$$(2\mu+\delta_k)\times\frac{4\ell^k}{3k!}<\frac{1}{2} \tag{36}$$

(see (22), (23), and (30)). Therefore, by Markov's inequality we have

$$\mathbb{P}(B_1\cup B_2\cup B_3)<\frac{1}{2}. \tag{37}$$

Next, we consider $B_4$. Here we use the abortion criteria for the iteration of Lemma 23, i.e., we use (26). By Lemma 27 we infer from (26) that

$$\left|\left\{K\in \mathrm{Cross}_k(\mathscr{Q}_J^{(1)})\colon\; \hat{Q}_J^{(k-1)}(K)\in \mathrm{MB}_{\mathscr{P}_J}(\mathscr{Q}_J,\eta/12)\right\}\right|\le\frac{2\varepsilon}{(\eta/12)^3}\binom{n}{k}.$$

Moreover, we say a $k$-tuple $K\in \mathrm{Cross}_k(\mathscr{Q}_J^{(1)})$ *misbehaves* if $|d(J^{(k)}|\hat{P}_J^{(k-1)}(K)-d(J^{(k)}|\hat{Q}_J^{(k-1)}(K))|>\eta/12$. Hence for every $\hat{Q}_J^{(k-1)}\notin \mathrm{MB}_{\mathscr{P}_J}(\mathscr{Q}_J,\frac{\eta}{12})$ it follows from the definition of $\mathrm{MB}_{\mathscr{P}_J}(\mathscr{Q}_J,\frac{\eta}{12})$ (see Definition 26) that

$$\left|\left\{K\in\mathcal{K}_k(\hat{Q}_J^{(k-1)})\colon\; K\text{ misbehaves}\right\}\right|\le\frac{\eta}{12}\left|\mathcal{K}_k(\hat{Q}_J^{(k-1)})\right|.$$

Therefore, the combination of the last two estimates yields

$$\left|\left\{K\in \mathrm{Cross}_k(\mathscr{Q}_J^{(1)})\colon\; K\text{ misbehaves}\right\}\right|\le\left(\frac{2\varepsilon}{(\eta/12)^3}+\frac{\eta}{12}\right)\binom{n}{k}\overset{(21)}{\le}\frac{\eta}{9}\binom{n}{k}.$$

Consequently, similar calculations as in (35) and (36) give that for randomly chosen $\ell$-set $L\in\Omega$ the expected number of misbehaved $k$-tuples $K\in\binom{L}{k}\cap \mathrm{Cross}_k(\mathscr{Q}_J^{(1)})$ is at most

$$\frac{\eta}{9}\times\frac{4\ell^k}{3k!}\overset{(20)}{\le}\frac{2\eta}{9}\binom{\ell}{k}, \tag{38}$$

since $\ell=|V(I^{(k)})|$ and $I^{(k)}$ is a hypergraph from the sequence $(H_i^{(k)})_{i=1}^{\infty}$.

Recalling that $B_4$ is the event that a random $\ell$-set $L\in\Omega$ contains more than $(4\eta/9)\binom{\ell}{k}$ misbehaved $k$-tuples we infer from (38) by Markov's inequality

$$\mathbb{P}(B_4)\le\frac{1}{2}. \tag{39}$$

From (37) and (39) we infer that there exist a "good" $\ell$-set, i.e., there exist an $\ell$-set $L \in \Omega \setminus (B_1 \cup \cdots \cup B_4)$. We now show that such an $\ell$-set has the desired properties (*L1-L3*) of Lemma 30.

First, since $L \notin B_1$ we have $L \in \mathrm{Cross}_\ell(\mathscr{P}_J^{(1)})$ as required. Moreover, (*L1*) holds by definition of $\Omega$ and (*L2*) is equivalent to $L \notin B_4$.

Finally, we focus on property (*L3*). Let a hypergraph $G^{(k)}$ with vertex set $L$ be given as in (*L3*). Let $\boldsymbol{P}_J(L) = \{P_J^{(j)}(L)\}_{j=1}^k$ be defined for $j \in [k]$ by

$$P_J^{(j)}(L) = \begin{cases} \bigcup \left\{ P_J^{(j)}(K) \colon\ K \in \binom{L}{k} \right\} & \text{if } j = 1 \dots, k-1\,, \\ \bigcup \left\{ J^{(k)} \cap \mathcal{K}_k(\hat{P}_J^{(k-1)}(K)) \colon\ K \in \binom{L}{k} \right\} & \text{if } j = k\,. \end{cases} \tag{40}$$

Since $L \notin B_2 \cup B_3$ we have for every $K \in \binom{L}{k}$ that $\boldsymbol{P}_J(K)$ is a $(\boldsymbol{\delta}(\boldsymbol{d}_J), \boldsymbol{d}_J, r(b_1, \boldsymbol{d}_J))$-regular $(n/b_1, k, k-1)$-complex and that $J^{(k)}$ is $(\delta_k, *, r(b_1, \boldsymbol{d}_J))$-regular w.r.t. $\hat{P}_J^{(k-1)}(K)$. Moreover, the assumptions on $G^{(k)}$ in (*L3*) imply $d(J^{(k)} | \hat{P}_J^{(k-1)}(K)) \geq \eta/12$ for $K \in G^{(k)}$ and $d(J^{(k)} | \hat{P}_J^{(k-1)}(K)) \leq 1 - \eta/12$ for $K \notin G^{(k)}$. Consequently, the definition of $\boldsymbol{P}_J(L)$ in (40) yields that $\boldsymbol{P}_J(L)$ is a $(\boldsymbol{\delta}', \geq \boldsymbol{d}', r(b_1, \boldsymbol{d}_J))$-regular, induced $(n/b_1, G^{(k)})$-complex with $\boldsymbol{\delta}' = (\boldsymbol{\delta}(\boldsymbol{d}_J), \delta_k)$ and $\boldsymbol{d}' = (\boldsymbol{d}_J, \eta/12)$. Due to the choice of $\delta_k$, and the functions $\delta_j$ and $r$ in (23), (24), and (25) and due to (33) we can apply the "induced" counting lemma, Corollary 15. It follows that $J^{(k)}$ contains at least

$$\frac{1}{2} \left( \frac{\eta}{12} \right)^{\binom{\ell}{k}} \prod_{j=2}^{k-1} d_{J,j}^{\binom{\ell}{j}} \times \left( \frac{n}{b_1} \right)^\ell \overset{(\mathrm{L}23.iv)}{\geq} \frac{1}{2} \left( \frac{\eta}{12} \right)^{\binom{\ell}{k}} \prod_{j=2}^{k-1} d_0^{\binom{\ell}{j}} \times \left( \frac{n}{b_1} \right)^\ell \overset{(31)}{=} \alpha |n|^\ell$$

induced copies of $G^{(k)}$. Then the choice of $J^{(k)}$ due to (32) implies that $G^{(k)} \notin \mathscr{F}$. Similarly, for every subset $L' \subseteq L$ we infer from Corollary 15 applied to $\boldsymbol{P}_J(L')$ that the number of induced copies of $G^{(k)}[L']$ in $J^{(k)}$ is at least

$$\frac{1}{2} \left( \frac{\eta}{12} \right)^{\binom{|L'|}{k}} \prod_{j=2}^{k-1} d_{J,j}^{\binom{|L'|}{j}} \times \left( \frac{n}{b_1} \right)^{|L'|} \overset{(\mathrm{L}23.iv)}{\geq} \frac{1}{2} \left( \frac{\eta}{12} \right)^{\binom{\ell}{k}} \prod_{j=2}^{k-1} d_0^{\binom{\ell}{j}} \times \frac{n^{|L'|}}{b_1^\ell} \overset{(31)}{=} \alpha |n|^{|L'|}\,.$$

Hence, the choice of $J^{(k)}$ in view of (32) implies $G^{(k)}[L'] \notin \mathscr{F}$. Since we inferred $G^{(k)}[L'] \notin \mathscr{F}$ for any $L' \subseteq L$ we have $G^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$, which is (*L3*) of Lemma 30. We thus showed that $L$ displays all required properties and this concludes the proof of Lemma 30. □

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