Super-Poincare algebra and Superfield formalism

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1 Poincare symmetry and spinors

The d-dimensional real vector space with metric

$$g_{\mu\nu} = diag(1, \underbrace{-1, \ldots, -1}_{d-1})$$

is called *Minkowski space*. A *Lorentz transformation*

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu \tag{1}$$

is an automorphism of Minkowski space which leaves the metric tensor invariant.

Lorentz transformation in 4-dimensions has six generators: three rotations J_i and three boosts K_i with following commutation relations:

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J_k \\ [K_i, K_j] &= -i\epsilon_{ijk}J_k \\ [J_i, K_j] &= i\epsilon_{ijk}K_j \end{aligned} \tag{2}$$

Let us define new generators as:

$$J^\pm_j = \frac{1}{2}(J_j \pm iK_j) \tag{3}$$

Note that these generators are the non-Hermitian. The commutation relations (2) then become

$$\begin{aligned} [J^\pm_i, J^\pm_j] &= i\epsilon_{ijk}J^\pm_k \\ [J^\pm_i, J^\mp_j] &= 0 \end{aligned}$$

We see that J^+ and J^- each generate a group $SU(2)$, and these two groups commute. One can easily show that the Lorentz group is then essentially $SU(2) \times SU(2) = SL(2, C)$ group.

The *Poincare group* contains Lorentz transformations and translations:

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{4}$$

The translations do not commute with the Lorentz transformations.

One rewrites Lorentz generators as $M_{0i} = K_i$ and $M_{ij} = \epsilon_{ijk}J_k$. Then the Poincare algebra becomes:

$$\begin{aligned} &[P_\mu, P_\nu] = 0 \\ &[M_{\mu\nu}, M_{\rho\sigma}] = ig_{\nu\rho}M_{\rho\sigma} - ig_{\mu\rho}M_{\nu\sigma} - ig_{\nu\sigma}M_{\mu\rho} + ig_{\mu\sigma}M_{\nu\rho} \\ &[M_{\mu\nu}, P_\rho] = -ig_{\rho\mu}P_\nu + ig_{\rho\nu}P_\mu \end{aligned} \tag{5}$$

The elements $M \in SL(2, C)$ are automorphisms of *spinor space*. Let ψ_α be an arbitrary element (called spinor) of the spinor space. Consider $SL(2, C)$–transformation of ψ_α:

$$\psi_\alpha \to \psi'_\alpha = M^\beta_\alpha \psi_\beta. \tag{6}$$

It is a fundamental representation of $SL(2,C)$. The conjugate representation is

$$\bar{\psi}_\alpha \to \bar{\psi}'_{\dot{\alpha}} = M^{*\beta}_{\dot{\alpha}} \psi_\beta. \tag{7}$$

We now want to enlarge the Poincare algebra by generators that transform either as undotted spinors Q^N_α or as dotted spinors $\bar{Q}^N_{\dot{\alpha}}$ under Lorentz transformations and that commute with translations:

$$\begin{aligned}
&[P_\mu, Q^N_\alpha] = 0 \\
&[P_\mu, \bar{Q}^N_{\dot{\alpha}}] = 0 \\
&[M_{\mu\nu}, Q^N_\alpha] = i(\sigma_{\mu\nu})^\beta_\alpha Q^N_\beta \\
&[M_{\mu\nu}, \bar{Q}^N_{\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}_{\dot{\alpha}} \bar{Q}^N_{\dot{\beta}}
\end{aligned} \tag{8}$$

The only possibility that algebra does not require extra generators is found to be the algebra:

$$\begin{aligned}
&\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2(\sigma_{mu})_{\alpha,\dot{\beta}} P^\mu \delta^{IJ} \\
&\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \\
&\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^*.
\end{aligned} \tag{9}$$

$Z^{IJ} = -Z^{JI}$ commute with all generators of SUSY algebra and called central charges. In N=1 SUSY there are no central charges.

The generators are described by the following representations of the Poincare algebra:

$$\begin{array}{ll}
P_\mu \to (1/2, 1/2) & M_{\mu\nu} \to (1,0) + (0,1) \\
Q_\alpha \to (1/2, 0) & \bar{Q}_{\dot{\alpha}} \to (0, 1/2).
\end{array}$$

2 Some properties

Since the full SUSY algebra contains the Poincare algebra as a subalgebra, any representation of the full SUSY algebra also gives a representation of the Poincare algebra. Each irreducible representation of the Poincare algebra corresponds to a particle. An irreducible representation of the SUSY algebra in general corresponds to several particles, in other words each superparticle, when viewed as irreducible representation of the SUSY algebra, is the direct sum of a collection of ordinary particles, called a multiplet.

Note that in systems where particles are created and annihilated the unitary representation of the Poincare algebra is not irreducible and for this reason there are some problems in Quantum Field Theory, which have been solved only partially, by procedures like renormalization.

There are some basic properties in SUSY:

1. All particles belonging to an irreducible representation of SUSY, i.e. within one supermultiplet, have the same mass.

2. The energy P_0 is always positive in SUSY.

3. A supermultiplet always contains an equal number of bosonic and fermionic degrees of freedom, i.e. the number n_b of bosons equals the number n_f of fermions.

Consider fermion number operator $(-)^{N_F}$ defined as

$$\begin{aligned}
&(-)^{N_F} |b\rangle = |b\rangle \\
&(-)^{N_F} |f\rangle = -|f\rangle .
\end{aligned} \tag{10}$$

$(-)^{N_F}$ anticommutes with Q:

$$(-)^{N_F}Q|f\,\rangle = (-)^{N_F}|b\rangle = |b>= Q|f\,\rangle = -Q(-)^{N_F}|f\rangle$$

Therefore

$$\{(-)^{N_F}, Q\} = 0. \tag{11}$$

Using the cyclicity of the trace one has

$$\begin{aligned} 0 = Tr\left(-Q(-)^{N_F}\bar{Q} + Q(-)^{N_F}\bar{Q}\right) &= Tr\left(-Q(-)^{N_F}\bar{Q} + (-)^{N_F}\bar{Q}Q\right) \\ &= Tr\left((-)^{N_F}\{Q,\bar{Q}\}\right) = Tr\left((-)^{N_F}2\sigma^\mu P_\mu\right) = 2\sigma^\mu p_\mu Tr\left((-)^{N_F}\right), \end{aligned} \tag{12}$$

where P_μ is replaced by its eigenvalues p_μ for the specific state. As result we get that

$$0 = Tr\left((-)^{N_F}\right) = \sum_{bosons}\langle b|(-)^{N_F}|b\rangle + \sum_{fermions}\langle f|(-)^{N_F}|f\rangle = \sum_{bosons}\langle b|b\rangle - \sum_{fermions}\langle f|f\rangle = n_b - n_f. \tag{13}$$

3 Superspace and Superfields

A very convenient way to obtain supermultiplets is the superfield technique. Superfield is a function on the superspace. Since the supercoordinates θ and $\bar{\theta}$ cannot be multiplied one by another more than some fixed number of times the Taylor expansion of a superfield over supercoordinates is finite.

Let us consider N=1 SUSY. An arbitrary scalar superfield can always be expended as:

$$F(x,\theta,\bar{\theta}) = f(x) + \theta\psi(x) + \bar{\theta}\bar{\xi}(x) + m(x)\theta\theta + n(x)\bar{\theta}\bar{\theta} + \theta\sigma^\mu\bar{\theta}v_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\rho(x) + d(x)\theta\theta\bar{\theta}\bar{\theta} \tag{14}$$

where $\theta = (\theta_a, \theta^{\dagger\dot{b}})^T$ and θ_a are Grassman.

We now want to realize the SUSY generators Q_α and its hermitian conjugate $\bar{Q}_{\dot{\alpha}} = (Q_\alpha)^\dagger$ as differential operators on superspace. We want that $i\epsilon^\alpha Q_\alpha$ generates a translation in θ^α by a constant infinitesimal spinor ϵ^α plus some translation in x^μ:

$$(1 + i\epsilon Q)F(x,\theta,\bar{\theta}) = F(x + \delta x, +\epsilon, \bar{\theta}) \tag{15}$$

By the ansatz we find that

$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha} + \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \tag{16}$$

Then the hermitian conjugate is

$$\bar{Q}_{\dot{\alpha}} = i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu. \tag{17}$$

We define SUSY variation as

$$\delta_{\epsilon,\bar{\epsilon}}F = (i\epsilon Q + i\bar{\epsilon}\bar{Q})F. \tag{18}$$

Let us find covariant derivatives D_α and $\bar{D}_{\dot{\alpha}}$ that commute with the SUSY generators Q and $\bar{Q}$. Then $\delta_{\epsilon\bar{\epsilon}}(D_\alpha F) = D_\alpha(\delta_{\epsilon\bar{\epsilon}}F)$ and the same for $\bar{D}_{\dot{\alpha}}$. One finds

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial^{\theta^\alpha}} + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu. \end{aligned} \tag{19}$$

A *chiral superfield* F is defined by the condition

$$\bar{D}_{\dot{\alpha}}F = 0 \tag{20}$$

and an *anti-chiral superfield* $\bar{F}$ by

$$D_\alpha\bar{F} = 0. \tag{21}$$

4 Supersymmetry in dimension $2+1$

Let us consider SUSY in the three-dimensional Minkowski space.

Consider scalar superfield $F(x,\theta,\bar{\theta})$:

$$F(x,\theta) = A(x) + \bar{\psi}\theta + \Phi\bar{\theta}\theta, \tag{22}$$

where $A(x)$ and Φ are scalar fields of the boson type and ψ and $\bar{\psi}$ are Majorana spinor fields of the fermionic type. Let us show that $\theta\bar{\theta}$ is invariant under the Lorentz transformations. We know that θ transforms under the Lorentz transformation as:

$$\theta' = \theta + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\theta \tag{23}$$

here $J^{\mu\nu} = -\frac{i}{4}[\sigma^\mu, \sigma^\nu]$. Now we can write transformations of θ^1 and θ^2:

$$\theta_1' = \theta_1 + \frac{i}{2}(\omega^{01}\theta_1 + \omega^{12}\theta_2 + \omega^{20}\theta_2) \tag{24}$$

$$\theta_2' = \theta_2 + \frac{i}{2}(-\omega^{01}\theta_2 - \omega^{12}\theta_1 + \omega^{20}\theta_1) \tag{25}$$

After some math one can easily show that:

$$\theta_1'\theta_2' = \theta_1\theta_2. \tag{26}$$

Since $\bar{\theta}\theta = 2\theta_1\theta_2$, $\bar{\theta}\theta$ is a Lorentz invariant. It is obvious that $\psi\bar{\theta}$ and $\theta\bar{\psi}$ are also invariants under the Lorentz transformations.

Under acting of the operator $\delta_{\epsilon\bar{\epsilon}}$ fields $A(x), \psi(x)$ and $\Phi(x)$ transform through each other:

$$\begin{aligned} A'(x) &= A(x) + 2\bar{\psi}\epsilon(x); \\ \psi'(x) &= \psi(x) - \sigma^\mu\epsilon\partial_\mu A - 2\Phi\epsilon; \\ \Phi'(x) &= \Phi(x) + \partial_\mu\bar{\psi}(x)\sigma^\mu\epsilon. \end{aligned} \tag{27}$$

The superfield $F(x,\theta,\bar{\theta})$ transforms as:

$$\delta_{\epsilon\bar{\epsilon}}F(x,\theta) = (\bar{\theta}\sigma^\mu\epsilon)\partial_\mu A(x) + 2\bar{\psi}(x)\epsilon + (\partial_\mu\bar{\psi}(x)\sigma^\mu\epsilon)\theta\bar{\theta} + 2\Phi(x)\bar{\theta}\epsilon. \tag{28}$$

We see that the operator $\delta_{\epsilon\bar{\epsilon}}$ from the odd part of the superalgebra mixes fields which transform in different ways under acting of the Poincare group. It mixes fermions and bosons.

Let us construct the simplest Lagrangian for the scalar superfield $F(x,\theta,\bar{\theta})$. We require that as equations of motion we obtain second-order differential equations. So the derivatives in the Lagrangian should occur no more than quadratically. We shal also require the Lagrangian would be invariant under Poincare transformations.

As far as our superspace is parametrised by usual coordinates x^μ and grassmanian coordinates θ, $\bar{\theta}$, one should integrate over all these coordinates to obtain translational invariant action. The action which would give us a Lorentz-invariant theory of free scalar and spinor fields has the following form:

$$S = \int d^3x d\bar{\theta} d\theta \left(\bar{D}FDF + MF^2\right) = \int d^3x\mathcal{L}. \tag{29}$$

Covariant derivation has the form:

$$D\Phi = -\frac{1}{2}\sigma^\mu\theta\partial_\mu A - \lambda + \frac{\theta\bar{\theta}}{2}\sigma^\mu\partial_\mu\psi - \theta\Phi. \tag{30}$$

One can show that the Lagrangian density is (terms with coefficent $\bar{\theta}\theta$):

$$\mathcal{L} = \left(-\frac{1}{2}(\partial_\mu A)^2 - \bar{\psi}\sigma^\mu\partial_\mu\psi + 2\Phi^2\right) + M(2A\Phi - \bar{\psi}\psi). \tag{31}$$

Let us find equation of motion for the field Φ:

$$2\Phi = -AM. \tag{32}$$

This field has no dynamics, but interaction of the fields A and λ with it gives rise to the mass terms. This can be easily seen if one put (32) to the action (29):

$$\mathcal{L} = -\frac{1}{2}\left((\partial_\mu A)^2 + M^2 A^2\right) - \bar{\psi}\left(\sigma^\mu \partial_\mu + M\right)\psi. \tag{33}$$

We see that fermion and boson fields in the scalar multiplet have the same mass.

References

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