The physical and mathematical background of the app:
single pipe flow

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Remark 1 *A detailed description of the mathematical computation to which I will refer in the later sections, can be found in [1]*

1 Applications background

The emission caused by vehicles is an import issue in these times. In order to reduce the pollutant emissions of CO, NO_x and C_xH_y in the exhaust gas, there are catalytic converters installed in the exhaust pipe system. The crucial quantity to control the efficiency of a catalytic converter is the temperature. Due to this reason, one is interested in how to ensure a sufficient high temperature in the catalytic converters in a short time after the engine start.

A special method of heating after the engine start is the combustion of unburnt gas in the catalytic converters. Modern cars can control the ratio of oxygen and fuel in the combustion chamber of the engine. By choosing a ratio with more fuel and less oxygen some unburnt gas gets to the catalytic converters and can be used there for an exothermic reaction.

2 Mathematical background

The mathematical model we use for the computation is the following system, consisting of two PDEs and 2 ODEs

$$
\begin{aligned}
\rho_t + (v + Q)\rho_x &= -q\rho, \\
z_t + (v + Q)z_x &= -\chi z K(T), \\
v_t = \frac{1}{\int_0^1 \rho dx} &\left[p_{1_l} - p_{1_r} - \int_0^1 \rho Q_t \ dx - \int_0^1 \rho(v+Q)q \ dx \right. \\
&\left. - C_f \int_0^1 \frac{\rho(v+Q)|v+Q|}{2} \ dx - C_c \int_0^1 \chi\rho(v+Q) \ dx \right], \\
(T_c)_t &= -h_c(T_c - T_{gas}).
\end{aligned}
$$

$$\text{(AM)}$$

The dynamic of the density ρ and the ratio of unburnt gas z are describes by the first two equations. The velocity u is now split into the space independent velocity component v and the aggregated energy balance Q. Lastly, the temperature of the catalytic converter is modelled by the last equation.

The letter χ denotes the locality of the catalyst, i.e. vanished outside the catalytic converter.

$$p_0 = \rho T$$

$$T_{gas} = \frac{1}{\int_0^1 \chi dx} \int_0^1 T(x,t)dx$$

$$T_{\text{Wall}} = \frac{1}{2}(T + T_{out})$$

$$u(x,t) = v(t) + Q[\rho, z](x,t) = v(t) + \int_0^x q[\rho, z](y,t)dy,$$

$$q[\rho, z] := \frac{1}{\gamma p_0} \left[-h(T - T_{\text{Wall}}) - \chi h_c(T - T_c) + \chi q_0 \rho z K(T) \right].$$

The model (AM) is the result of a low Mach number limit, executed on the Euler equation of gas dynamics (for details see [1]). This limit is performed by expanding asymptotically the pressure p. This is the reason, why we have to differentiate via indices the order of the term of the asymptotic expansion. The index of the pressure p, denotes the order of the term of the asymptotic expansion, which lead to the model (AM).

You will find a complete list of the parameters and their definition in the appendix.

3   Numerical treatment

The numerical algorithm we use is a combination of the explicit upwind method (for the PDEs) and the explicit Euler method (for the ODEs).

1. Compute from the CFL condition the right step size in time Δt.

2. Use the explicit upwind method to compute ρ and z for the next time step.

3. Use the explicit Euler rule to compute v and T_c for the next time step.

4. Go back to 1.

References

[1] I. Gasser and M. Rybicki. *Modelling and simulation of gas dynamics in an exhaust pipe*, Applied Mathematical Modelling, (2012)

[2]   M. Rybicki and R. Krenzler   Single Pipe Flow App   Visit website: http://www.math.uni-hamburg.de/home/rybicki/apps.

A   Parameters

$$\gamma M^2 = \frac{\rho_r u_r^2}{p_r} \qquad\qquad \gamma = \frac{c_p}{c_v} \qquad \gamma - 1 = \frac{R}{c_v} \qquad C_f := \frac{\xi x_r}{\tilde{d}}$$

$$C_c := \frac{\tilde{C}_c x_r}{u_r} \qquad\qquad h := \frac{4\tilde{h} x_r}{\tilde{d}\rho_r u_r c_v} \qquad h_c := \frac{\tilde{h}_c x_r}{\rho_r u_r c_v} \qquad q_0 := \frac{\rho_r z_r \tilde{q}_0 R}{p_r c_v}$$

$$K(T) := \frac{\tilde{K}_0 x_r}{u_r} \exp(-\frac{E^+}{T_r T})$$

Parameter	Name	Unit	Parameter value
ξ	wall friction		$4.52 \cdot 10^{-3}$
$\tilde{C}_c$	cc friction	s^{-1}	800
$\tilde{q}_0$	heat release	$\text{m}^2\ \text{s}^{-2}$	$5 \cdot 10^6$
$\tilde{h}$	heat transfer wall	$\text{kg s}^{-3}\ \text{K}^{-1}$	150
$\tilde{h}_c$	heat transfer cc	$\text{kg s}^{-3}\ \text{K}^{-1}$	100
$\tilde{K}_0$	reaction rate	s^{-1}	100
$\tilde{E}^+$	activation energy	K	600
γ	adiabatic exponent		1.4
R	Boltzmann constant	$\text{m}^2\ \text{s}^{-2}\ \text{K}^{-1}$	287.08
c_v	heat capacity	$\text{m}^2\ \text{s}^{-2}\ \text{K}^{-1}$	717.7
$\tilde{T}_{out}$	external temperature	K	290.28

Table 1: Parameters, units, parameter values

Quantity	Unit	Reference quantity	Reference value
$\tilde{t}$	s	$t_r = x_r/u_r$	0.1s
$\tilde{x}$	m	$x_r = \tilde{L}$	1m
$\tilde{\rho}$	kg m^{-3}	ρ_r	1.2 kg m^{-3}
$\tilde{u}$	m s^{-1}	u_r	10 m s^{-1}
$\tilde{p}$	$\text{kg m}^{-1}\ \text{s}^{-2}$	p_r	10^5 kg m^{-1} s^{-2}
$\tilde{T}$	K	$T_r = p_r/(R\rho_r)$	300 K
$\tilde{z}$		z_r	0.1

Table 2: Quantities, units, reference quantities, reference values