Global gauge anomalies in coset models of conformal field theory

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Outline

1. Global gauge anomalies
   - Gauged Wess-Zumino-Witten models
   - Gauge invariance obstructions
   - No-anomaly condition in coset models

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   - Gauged Wess-Zumino-Witten models
   - Gauge invariance obstructions
   - No-anomaly condition in coset models

2. Classification of anomalies
   - Case $\mathfrak{h} = \mathfrak{g}$
   - Regular subalgebras
   - R- and S- subalgebras

Gauged Wess-Zumino-Witten models

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- In the following, $\omega = Id$ (untwisted) to start with

Coset models

Coset models

Let $H \subseteq \tilde{G}$ and $\mathrm{Lie}(H) = \mathfrak{h} \subseteq \mathfrak{g}$

Coset model

$$\int \mathcal{D}g\, \mathcal{D}A\; \mathrm{e}^{\mathrm{i}S[g,A]} \quad \text{with} \quad \begin{array}{ll} g : \Sigma \to G = \tilde{G}/Z & Z \equiv \mathcal{Z}(G) \\ A : \mathfrak{h}\text{-valued 1-forms on } \Sigma & \end{array}$$

Gauge invariance

Gauge invariance

- Gauge invariance of the Feynman amplitudes

$$\exp\left[iS({}^h g, {}^h A)\right] \;=\; \exp\left[iS(g, A)\right]$$

for

$$h : \Sigma \to H/(H \cap \tilde{Z}), \qquad {}^h g \;=\; hg\, h^{-1}\,, \qquad {}^h A \;=\; hAh^{-1} + hdh^{-1}$$

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- Always true for "small" gauge transformations (= homotopic to unity).
- For "large" gauge transformations, topological obstructions to gauge invariance can occur → **global gauge anomaly**
- ⇒ Destructive interferences in the coset model partition function which lead to inconsistent theories.

[Ref] K. Gawędzki, R. R. Suszek, K. Waldorf, *Global gauge anomalies in two-dimensional bosonic sigma models*, Commun. Math. Phys. **302** (2011)

No-anomaly condition in coset models

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Using $B = \frac{k}{4\pi}\mathrm{tr}\left(g^{-1}dg(h^{-1}dh) + (dg)g^{-1}h^{-1}dh + g^{-1}(h^{-1}dh)g(h^{-1}dh)\right)\Big]$

$$\frac{\exp\left[iS^{WZ}({}^{h}g)\right]}{\exp\left[iS^{WZ}(g) + \mathrm{i}\int_{\Sigma}(g,h)^{*}B\right]} = 1$$

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It is enough to check it for $\Sigma = S^1 \times S^1$ and

$$h(\mathrm{e}^{\mathrm{i}\sigma_1}, \mathrm{e}^{\mathrm{i}\sigma_2}) = \mathrm{e}^{\mathrm{i}\sigma_1 \tilde{M}}\,, \qquad g(\mathrm{e}^{\mathrm{i}\sigma_1}, \mathrm{e}^{\mathrm{i}\sigma_2}) = \mathrm{e}^{\mathrm{i}\sigma_2 M}$$

for $M, \tilde{M} \in P^\vee(\mathfrak{g})$ such that

$$\tilde{z} \equiv e^{2\mathrm{i}\pi\tilde{M}} \in H \cap \tilde{Z} \quad \text{and} \quad z \equiv e^{2\mathrm{i}\pi M} \in Z\,.$$

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No anomaly condition

$$k\,\mathrm{tr}(M\tilde{M}) \in \mathbb{Z} \qquad \forall M, \tilde{M} \in P^\vee(\mathfrak{g}), \qquad z \in Z, \tilde{z} \in H \cap \tilde{Z}$$

Equivariant gerbes interpretation

Feynman amplitudes :

$$\exp\left[iS^{WZ}(g)\right] = \mathrm{Hol}_{\mathcal{G}_k}(g)$$

where $\mathcal{G}_k$ is a bundle gerbe with unitary connection over G

- Gauge invariance $\Leftrightarrow$ Existence of a H-equivariant structure on $\mathcal{G}_k$

Remark : Let J_z be the WZW simple current associated to $z \in \tilde{Z}$, then no-anomaly condition $\Leftrightarrow$ monodromy charge $Q_{J_z}(J_{\tilde{z}}) \in \mathbb{Z}$ for $z \in Z, \tilde{z} \in H \cap \tilde{Z}$

Classification of anomalies

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No-anomaly condition

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- Specify $\mathfrak{g}$ **simple** Lie algebra (using Dynkin diagram classification) $\rightarrow \tilde{G}$
- Specify $Z \subseteq \tilde{Z} \rightarrow G = \tilde{G}/Z$
  Remark : If $G = \tilde{G}$ ($Z = \{1\}$), there is no global gauge anomaly.
- Specify $\mathfrak{h} \subset \mathfrak{g}$ **semisimple** subalgebra $\rightarrow H \subseteq \tilde{G}$
- Compute $k\ \mathrm{tr}(M\tilde{M})$

Classification = forbidden levels k corresponding to anomalous models ($k\ \mathrm{tr}(M\tilde{M}) \notin \mathbb{Z}$)

The easy part

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- For $\mathfrak{g} = \mathfrak{g}_2,\ \mathfrak{f}_4,$ and $\mathfrak{e}_8$, the centre is trivial : $\tilde{Z} = \mathcal{Z}(\tilde{G}) = \{1\}$.
- For $\mathfrak{g} = B_r,\ C_r,$ and $\mathfrak{e}_7$, restrictions on k for the absence of anomaly follow for the admissible levels of the underlying WZW model.

Proposition

The coset models corresponding to Lie algebras $\mathfrak{g} = B_r,\ C_r,\ \mathfrak{g}_2,\ \mathfrak{f}_4,\ \mathfrak{e}_7,$ or $\mathfrak{e}_8$ and any subalgebra $\mathfrak{h}$ do not have global gauge anomalies.

Conclusion : Anomalies occur only for $\mathfrak{g} = A_r, D_r$ or $\mathfrak{e}_6$
(N.B. : they are all simply laced)

Case $\mathfrak{h} = \mathfrak{g}$: the example of $\mathfrak{e}_6$

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- $\tilde{Z} \cong \mathbb{Z}_3$, the only non trivial subgroup is $Z = \tilde{Z}$
- $M, \tilde{M} \in P^\vee(\mathfrak{e}_6)$ can be described in a subspace of $\mathbb{R}^7$ such that

$$M = \left(\frac{a}{6}, \dots, \frac{a}{6}, \frac{-5a}{6}, \frac{a}{\sqrt{2}}\right) + q \qquad a \in \mathbb{Z},\ q \in Q^\vee(\mathfrak{e}_6)$$
$$\tilde{M} = \left(\frac{\tilde{a}}{6}, \dots, \frac{\tilde{a}}{6}, \frac{-5\tilde{a}}{6}, \frac{\tilde{a}}{\sqrt{2}}\right) + \tilde{q} \qquad \tilde{a} \in \mathbb{Z},\ \tilde{q} \in Q^\vee(\mathfrak{e}_6).$$

- No anomaly if $\quad k\ \mathrm{tr}(M\tilde{M}) = k\dfrac{4a\tilde{a}}{3} \quad$ is an integer.

Proposition

The coset models corresponding to Lie algebra $\mathfrak{g} = \mathfrak{e}_6$, $Z = \mathbb{Z}_3$ and subalgebra $\mathfrak{h} = \mathfrak{g}$ do not have global gauge anomalies iff $k \in 3\mathbb{Z}$.

Case $\mathfrak{h} = \mathfrak{g}$: A_r and D_r

Case $\mathfrak{h} = \mathfrak{g}$: A_r and D_r

- Same kind of computation for A_r and D_r $\rightarrow$ full classification of the anomalous levels k for $\mathfrak{h} = \mathfrak{g}$.
  Remark : the centre corresponding to D_r depends on the parity of r ($\cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$) $\rightarrow$ different anomaly condition.

- Proper subalgebras $\mathfrak{h} \subset \mathfrak{g}$ will relax (or not) the constrains on k for the non anomalous models by imposing some restriction on $\tilde{M}$ in $k\ \mathrm{tr}(M\tilde{M}) \in \mathbb{Z}$

Regular subalgebras

Regular subalgebras

Canonical decomposition of $\mathfrak{g}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}} \oplus \left(\bigoplus_{\alpha \in \Delta_{\mathfrak{g}}} \mathbb{C} e_{\alpha} \right)$$

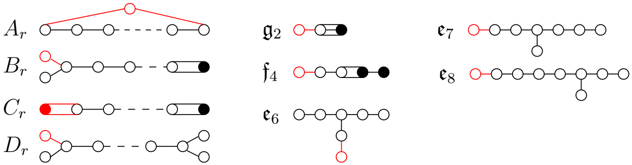
The subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is regular and semisimple if

$$\mathfrak{h}^{\mathbb{C}} = \mathfrak{t}_{\mathfrak{h}}^{\mathbb{C}} \oplus \left(\bigoplus_{\alpha \in \Delta_{\mathfrak{h}} \subset \Delta_{\mathfrak{g}}} \mathbb{C} e_{\alpha} \right)$$

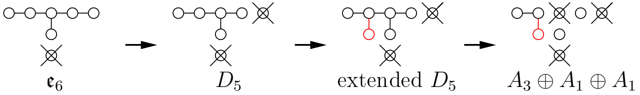
where $\mathfrak{t}_{\mathfrak{h}} \subset \mathfrak{t}_{\mathfrak{g}}$ is a Cartan subalgebra of $\mathfrak{h}$ and

- $\alpha \in \Delta_{\mathfrak{h}} \Rightarrow -\alpha \in \Delta_{\mathfrak{h}}$
- $\alpha \in \Delta_{\mathfrak{h}}$ span $\mathfrak{t}_{\mathfrak{h}}^{\mathbb{C}}$

Take the extended Dynkin Diagram of $\mathfrak{g}$ ($\delta = -\phi$ is the lowest root)



Remove one or several roots and take the extension of the subdiagrams.



Ex : $\mathfrak{e}_6$, D_5 and $A_3 \oplus 2A_1$ are regular subalgebras of $\mathfrak{e}_6$.

Regular subalgebras of classical algebras

- Regular subalgebras of A_r :

$$\mathfrak{h} = A_{r_1} \oplus \ldots \oplus A_{r_m}, \qquad \sum_{i=1}^{m} (r_i + 1) \leq r + 1$$

- Regular subalgebras of D_r :

$$\mathfrak{h} = A_{r_1} \oplus \ldots \oplus A_{r_m} \oplus D_{s_1} \oplus \ldots \oplus D_{s_n}, \qquad \sum_{i=1}^{m} (r_i + 1) + \sum_{i=1}^{n} s_i \leq r$$

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Requiring $\mathrm{e}^{2\pi \mathrm{i} \tilde{M}} \in H$ for each $\mathfrak{h}$ in $k \, \mathrm{tr}(M\tilde{M}) \in \mathbb{Z}$ leads to a specific condition on k without anomalies.
Ex : Anomalies disappear if the rank inequality is strict.

Regular subalgebras of classical algebras

- Protection property : $\mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \mathfrak{g}$ regulars with $\mathfrak{h}_2$ leading to non anomalous model $\Rightarrow \mathfrak{h}_1$ is also a non anomalous model.
- Subalgebras related by inner or outer automorphism, or by Weyl group transformations : same anomaly problem.

One obtains a full classification of the anomalous models for regular subalgebras $\mathfrak{h}$ of classical algebras

Regular subalgebras of $\mathfrak{e}_6$

Proposition

Coset models built with $\mathfrak{h}$ regular subalgebra of $\mathfrak{e}_6 \to$ no global gauge anomaly for $k \in \mathbb{Z}$, except for

$$\left.\begin{array}{l} \text{Rank } 6 : \mathfrak{e}_6, A_5 \oplus A_1, A_2 \oplus A_2 \oplus A_2 \\ \text{Rank } 5 : A_5, A_2 \oplus A_2 \oplus A_1 \\ \text{Rank } 4 : A_2 \oplus A_2 \end{array}\right\} k \in 3\mathbb{Z}$$

The other models are anomalous.

R- and S- subalgebras

R- and S- subalgebras

$$\mathcal{R}(\mathfrak{h}) := \min_{\mathfrak{m}} \{\mathfrak{m} \subset \mathfrak{g} \mid \quad \mathfrak{h} \subset \mathfrak{m} \subset \mathfrak{g} \text{ and } \mathfrak{m} \text{ regular.}\}$$

- $\mathcal{R}(\mathfrak{h}) = \mathfrak{g} \Rightarrow \mathfrak{h}$ is an S-algebra
- $\mathcal{R}(\mathfrak{h}) \subsetneq \mathfrak{g} \Rightarrow \mathfrak{h}$ is an R-algebra

Different subalgebras $\mathfrak{h}$ may be isomorphic as algebras.

Ex : $\mathfrak{h} \cong A_1 \subset \mathfrak{e}_6$ has 1 regular embedding, 2 S- and 17 R- embeddings.

R- and S- subalgebras of $\mathfrak{e}_6$

- Explicit embedding is needed in the no-anomaly condition to find $\tilde{z} \in H$.
- Equivalent embeddings leads to the same anomaly problem.
- There exists inequivalent embeddings for the same $\mathfrak{h}$ and $\mathcal{R}(\mathfrak{h})$

$\Rightarrow$ Full classification of the anomalous coset models for $\mathfrak{g} = \mathfrak{e}_6$ and arbitrary semisimple nonregular subalgebra $\mathfrak{h}$

R- and S- subalgebras : classical case

- No explicit embedding classification for $\mathfrak{g} = A_r$ or D_r with arbitrary r.

- No conceptual difficulty to find the anomalous models for a fixed r, but no general result for arbitrary r as for the regular case.

Let's twist the coset models

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- Twisted gauge invariance : $g \to hg\omega(h)^{-1}$
- Inequivalent twisted models only for $\omega \in \mathrm{Out}(G)$ / $\mathrm{Inn}(G) \cong$ Dynkin diagram symmetries of $\mathfrak{g}$
- At most one non trivial ω except for $D_r, r > 4$ even (2 automorphisms) and D_4 (3 automorphisms)
- No anomaly condition becomes :

$$\underbrace{c_{\tilde{z}\omega(\tilde{z})^{-1},z}}_{\in Hom(Z\otimes Z,U(1))} \quad \exp\left[-2i\pi k\, \mathrm{tr}(M\omega(\tilde{M}))\right] = 1$$

where

$$\tilde{z} \equiv e^{2i\pi\tilde{M}} \in Z^{\omega} \cap \tilde{H} \quad \text{and} \quad z \equiv e^{2i\pi M} \in Z\,.$$

- The classification of anomalous twisted models computed the same way as for the untwisted case.

Summarizing

Classification of the anomalous models (twisted and untwisted)

List of forbidden levels k for coset models built with :

- a simple classical Lie algebra $\mathfrak{g}$ and its semisimple regular subalgebras $\mathfrak{h}$
- Lie algebra $\mathfrak{e}_6$ and arbitrary semisimple subalgebra $\mathfrak{h}$

Remark : Anomalies occur only for $\mathfrak{g} = A_r$, D_r and $\mathfrak{e}_6$ in the untwisted case, and for $\mathfrak{g} = D_r$, r even in the twisted case.

[Ref] P. de Fromont, K. Gawędzki, C. Tauber, *Global gauge anomalies in coset models of conformal field theory* (arXiv :1301.2517)

Prospects

- Global gauge anomalies in T-dual models ?
- Global gauge anomalies in supersymmetric coset models ?

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Thank you.