

# TWENTY-FIVE YEARS OF TWO-DIMENSIONAL RATIONAL CONFORMAL FIELD THEORY

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## 1 Introduction

In this article we try to give a condensed panoramic view of the development of two-dimensional rational conformal field theory in the last twenty-five years. Given its limited length, our contribution can be, at best, a collection of pointers to the literature. Needless to say, the exposition is highly biased, by the taste and the limited knowledge of the authors. We present in advance our apologies to everyone whose work has been inappropriately represented or even omitted.

The study of conformal field theories in two dimensions has its origin in several distinct areas of physics:

- In the attempt to describe the strong interactions in elementary particle physics in the framework of dual models.
- Under the label string theory, these models were reinterpreted as a perturbation series containing a gravitational sector. Conformal field theories appear as theories defined on the world sheet that is swept out by the string [56].
- In statistical mechanics, conformal field theory plays a fundamental role in the theory of two-dimensional critical systems [192, 92] by describing fixed points of the renormalization group.

- Conformal field theories on surfaces with boundary arise in quasi one-dimensional condensed matter physics in the description of impurities, e.g. in the Kondo effect [1].
- Chiral conformal field theories describe universality classes of quantum Hall fluids [97]; conformal blocks are used as approximate wave functions for quasi-particles [173].

In the last 25 years, two-dimensional rational conformal field theory has been moreover a major area of cross-fertilization between mathematics and physics. Many fields of mathematics have both benefitted and contributed to conformal field theory. Examples include representation theory, infinite-dimensional algebra, the theory of modular forms, algebraic and differential geometry, and algebraic topology.

We find it helpful to divide the era we are discussing into three periods. The first is what we will refer to as the classical period, comprising roughly the second half of the 1980s, in which much fundamental insight into conformal field theories was obtained. This was followed by a period of subsequent consolidation, which essentially comprises the first half of the nineties. The third period started basically with advent of D-branes in string theory, which had a significant impact on the field: it gave a strong boost to the study of boundary conditions, which resulted in a much deeper understanding of the structure of rational conformal field theory. This period of research has brought (higher) category theoretic structures to the forefront, both in algebraic and in geometric approaches to rational conformal field theory. We end with a discussion of some topics of current interest and with a list of omissions.

There is a close relation between conformal field theory in two dimensions and topological field theory in three dimensions. In fact, both for chiral conformal field theory and for full local conformal field theory, many structural aspects become much more transparent by using insights obtained from three-dimensional topology. For this reason, some of the developments in three-dimensional topological field theory are covered in this review as well.

## 2 Groundbreaking insights

Historically, the classical period has its roots in the theory of dual resonance models in the early seventies and in the study of operator product expansions in quantum field theory. Important pieces of insight gained already back then are the following:

- Chiral quantities – in particular conserved chiral currents – live on a double cover  $\widehat{X}$  of the surface  $X$  on which the conformal field theory is considered [6]. The surface  $\widehat{X}$  is naturally oriented and inherits a conformal structure from  $X$ . In two dimensions, these data are equivalent to a *holomorphic* structure on  $\widehat{X}$ . This relationship is at the basis of powerful complex-analytic techniques available for the study of two-dimensional conformal field theories.
- The requirement of conformal invariance yields strong constraints on the structure of quantum field theory. For instance, when combined with the principle of locality, it leads to the Lüscher-Mack theorem [165, 111] which gives the most general commutation relations among the components of the stress-energy tensor. The modes of the stress-energy tensor are generators of an infinite-dimensional Lie algebra, the Virasoro algebra. This forms the basis for the use of techniques from infinite-dimensional algebra in conformal field theory.
- Operator product expansions can be studied with the help of representation theoretic methods see e.g. [58, 199]. Indeed, conformal field theory in general has been one of the driving forces for the representation theory of infinite-dimensional Lie algebras and related structures.

- Current symmetries induce conformal symmetries, via a variant of the Sugawara construction [18, 48, 215].
- Low-dimensional field theories allow for ‘exotic’ statistical behavior of quantum fields more general than the one of bosons or fermions [221, 158], now referred to as anyon statistics. Also, bosonic theories can have sectors which behave like fermions, and vice versa [44, 94, 87]; this is known as the boson-fermion correspondence.

The 1980s have witnessed impressive progress in two-dimensional conformal field theories. To give an idea of these developments, we refer to classics in the literature and, where appropriate, indicate on which previous developments the respective article is based. A list of highlights is, in chronological order:

- In 1984, Belavin, Polyakov and Zamolodchikov [24] used infinite-dimensional symmetries to reduce the description of operator algebra expansions to a finite-dimensional problem and computed the OPE coefficients for bulk fields in Virasoro minimal models. Chiral symmetry structures that are strong enough to allow for such a reduction from infinite- to finite-dimensional problems are one of the hallmarks of *rational* conformal field theories.

The work of [24] strongly relies on results from the study of dual models, in particular about the structure of the conformal group and of the Virasoro algebra. One crucial input is information about highest weight representations of the Virasoro algebra from the work of Feigin and Fuks [70] and Kac [144].

- Also in 1984, Witten [236] introduced a Lagrangian formulation of conformal field theories with current algebra symmetries, now known as Wess-Zumino-Witten models. This takes the form of a sigma model with target space a group manifold, as in the previously studied principal chiral model (see e.g. [51, 184]), but includes a Wess-Zumino term, i.e. [235, 183] a multi-valued topological action.
- In the same year, Knizhnik and Zamolodchikov [153] derived differential equations on the correlators of the same class of models by studying combined null vectors of the affine Lie algebra and the Virasoro algebra.
- In 1986, Goddard, Kent and Olive used the coset construction [129] to realize the chiral unitary Virasoro minimal models. The coset construction is, even to date, a major source for rational chiral conformal field theories. In addition, it supplies examples of quantum field theories with gauge symmetries that are at the same time exactly tractable using methods from representation theory.
- Also in 1986, Borchers [30] introduced the notion of a vertex algebra to mathematics as a formalization of the physical concept of a chiral algebra, i.e. the algebraic structure encoding the chiral symmetries of a conformal field theory. His motivation was to gain a better understanding of “monstrous moonshine”, a conjecture connecting the largest sporadic group, the monster group, to modular forms. A vertex algebra can be thought of as a generalization of a  $\mathbb{Z}_+$ -graded commutative associative unital algebra equipped with a derivation of degree 1 with a multiplication depending on a formal coordinate. Various concepts from conformal field theory and string theory are instrumental for Borchers’ work. In particular, the no-ghost theorem of Goddard and Thorn [130] ensures that the Lie algebras associated to the vertex algebras of interest are indeed generalized Kac-Moody algebras. The physical principle of locality appears in the definition of vertex algebras as a requirement on the commutator of chiral fields.

- And again in 1986, Cardy [37] showed that imposing modular invariance of the torus partition function constitutes a powerful constraint on the field content of a conformal field theory. One year later, Cappelli, Itzykson and Zuber [36] presented a classification of modular invariant partition functions for the  $\mathfrak{su}(2)$  WZW model and for Virasoro minimal models. The modular properties of characters of affine Lie algebras computed by Kac and Peterson [146] are crucial input. These classifications display an A-D-E pattern similar to the classification of simply laced complex simple Lie algebras, of discrete subgroups of  $SU(2)$ , or of elementary catastrophes.
- In 1987 Fröhlich [95, 96] emphasized the relevance of the braid group for the statistical properties of quantum fields in two and three dimensions, and their relations with monodromy properties of correlation functions and with the Yang-Baxter equation. Closely related observations about statistical properties and the exchange relations for chiral fields were made by Rehren and Schroer [195], Tsuchiya and Kanie [225], and Longo [161]. Braid group statistics includes anyon statistics as a special case, namely the one in which the braid group representation is one-dimensional.
- In 1988-1989 Moore and Seiberg [175, 176, 177] unraveled much of the categorical structure underlying chiral rational conformal field theories. Their work was based on the results about braid group statistics just mentioned and on the idea of Friedan-Shenker [93] to formulate two-dimensional conformal field theory as analytic geometry on a universal moduli space of Riemann surfaces. It extended results about the fusion rules that had been obtained by E. Verlinde [229] using heuristic arguments about the factorization of conformal blocks on elliptic curves. Felder, Fröhlich and Keller [76] introduced a family of coproducts on chiral algebras to gain a representation theoretic understanding of these findings.
- Also in 1989, Fredenhagen, Rehren and Schroer [80] discussed the superselection structure of sectors with braid group statistics in low-dimensional quantum field theory. The aspects particular two two-dimensional *conformal* field theory were analyzed in [81, 112]. These studies are performed in the algebraic framework for quantum field theory, which was developed by Doplicher, Haag and Roberts [60, 61] and others and is, in turn, based on earlier work of Haag and Kastler [132]. A crucial role in the algebraic framework is played by the locality principle, which provides a conceptually clear understanding of the appearance of braid group statistics in low dimensional quantum field theory.
- At the same time, Witten [237] developed non-abelian Chern-Simons theory, a three-dimensional topological field theory which yields invariants for links that can be specialized to the Jones polynomial. The Jones polynomial can equally be recovered from differential equations on expectation values of Wilson lines that can be derived in a functional integral approach, see [100]. Witten's results generalize in particular earlier work by A.S. Schwarz [210] on abelian gauge theories having a Chern-Simons term only. Witten also shows how structure of chiral rational conformal field theory, e.g. the Verlinde formula for the fusion rules, becomes transparent when viewed from three-dimensional geometry.

The Chern-Simons three-form is closely connected to characteristic classes. When Chern-Simons theory is evaluated on special three-manifolds, it provides invariants of knots and links. Thereby Chern-Simons theory provides several links between conformal field theory and algebraic topology.

At about the same time, Atiyah presented an axiomatization of topological quantum field theory [12] that is based on a Segal's proposal [213] for an axiomatics of conformal field theory.

- Again in 1989, Cardy [38] exhibited a close relation between the Verlinde formula and boundary conditions of a rational conformal field theory with modular invariant partition function given by charge conjugation.

Numerous reviews and books provide details about the work in the classical period. An (again biased) selection is [111, 8, 178, 52, 204, 113].

### 3 Extension and consolidation

The papers listed in section 2 continue to stimulate much of the ongoing research in the field, both in mathematics and in physics. Here we present a very brief summary of some important subsequent developments.

Subsection 3.1 gives an overview of chiral symmetry structures. An important aspect of the representation theory of these structures is that they allow the construction of vector bundles with connection on the moduli space of complex curves, called conformal blocks; aspects of this theory are discussed in subsection 3.2. The system of vector bundles formed by the conformal blocks is essentially the mathematical formalization of chiral conformal field theory. This is a highly non-trivial system of bundles; it allows one to endow the representation category of the chiral symmetry algebra with additional structure. In the strongest possible situation, one obtains the structure of a modular tensor category, which is the subject of subsection 3.3.

Full local conformal field theories – which arise e.g. as the quantization of certain Lagrangian field theories, or in the continuum limit of certain lattice models – can be constructed from chiral conformal field theories. A specific ingredient of this construction is to find a modular invariant torus partition function. This is the subject of subsection 3.4. Lagrangian approaches both to conformal field theory and to topological field theory are very briefly discussed in subsections 3.5 and 3.6, respectively.

#### 3.1 Chiral symmetry structures

As a first important example of a vertex algebra, Frenkel, Lepowsky and Meurman [89] constructed the monster vertex algebra, whose group of vertex algebra automorphisms is the monster sporadic group. This vertex algebra plays a crucial role in Borcherds' proof of the moonshine conjecture (for reviews, see e.g. [118, 128]). The monster vertex algebra is actually a conformal vertex algebra, i.e. it comes with a Virasoro element, which gives rise to a field whose modes furnish a representation of the Virasoro algebra on any module over the vertex algebra.

Much activity was aimed at constructing further classes of vertex algebras or chiral symmetry algebras in different formalizations. Some major lines of development were the following.

- A strong driving force was the quest for larger symmetry structures which, while having Virasoro central charge larger than one, still yield rational theories. One of the resulting notions is the structure of a  $\mathcal{W}$ -algebra; for a review of such algebras we refer to [31].
- By general arguments, every modular invariant partition function can be constructed in two steps [177]: in the first step the chiral algebras for left- and right-moving degrees of freedom are extended, yielding chiral theories with isomorphic fusion rules. The pairing of left- and right-movers in the local theory then requires one to select an isomorphism, and this choice constitutes the second step. Extensions of chiral symmetry algebras are therefore of much interest. Two classes of extensions are particularly well understood: conformal embeddings

[15, 205] of chiral algebras, which are generated by non-abelian currents, and extensions by representations of quantum dimension one, so-called simple currents [207, 16, 59].

- The original construction of the monster vertex algebra involved the extension of an orbifold of some lattice vertex algebra involving the Leech lattice. The orbifold construction [57] continues to be a rich source of algebraic [53] and geometric [134] constructions. It turns out that the relevant category of world sheets for an orbifold theory with orbifold group  $G$  are  $G$ -coverings, which can have branch points at the insertions of twist fields. The product of  $N$  copies of a vertex algebra carries an obvious action of any subgroup  $G \subseteq \mathfrak{S}_N$ . The corresponding orbifold theory is called a permutation orbifold. Permutation orbifolds enter crucially in a proof of the congruence subgroup conjecture [17].
- The coset construction [129] provides a particularly important generalization of vertex algebras based on affine Lie algebras. It also admits a Lagrangian description [123] and continues to be an important testing ground for new concepts in conformal field theory. For most coset conformal field theories the problem of selection rules and field identification, and of resolving fixed points under this identification, arises. In spite of progress on the representation theory of the coset chiral algebra, both in terms of the underlying affine Lie algebras [107] and concerning category theoretic aspects [98], this problem is not completely solved. Irrational conformal field theories [133] provide a hint to a large generalization of the coset construction; for the time being, these theories offer, however, more challenges than results.

On the other hand, the classification of vertex algebras, or just of rational conformal vertex algebras, is not a realistic goal. For instance, every even lattice, in particular every even self-dual one, provides such a vertex algebra. Even self-dual lattices only exist in dimensions  $d$  which are a multiple of 8. It follows from the Siegel-Minkowski mass formula that already in dimension 32 there are at least  $10^7$  such lattices. The corresponding conformal vertex algebras have a single irreducible representation and Virasoro central charge equal to  $d$ . More generally, conformal vertex algebras with a single irreducible representation must have a Virasoro central charge  $c$  which is a multiple of 8. For  $c=24$  so-called meromorphic conformal field theories, which are supposed to correspond to such vertex algebras, were classified in [203].

The vertex algebras relevant for *rational* conformal field theory have a semisimple representation category. In this case various aspects of the representation category are well understood, including in particular the Huang-Lepowsky theory of tensor products for categories of modules over a conformal vertex algebra (for a review see [140]). Much of the work of the Huang-Lepowsky school is built on this theory. A related approach to tensor products is given by the Gaberdiel-Kausch-Nahm algorithm [114, 180]. An extension of tensor product theory to the non-semisimple case was developed in [141].

For several reasons, tensor categories associated with affine Lie algebras are of particular interest: they are related to problems in algebraic geometry, and they have strong connections with representation categories of quantum groups. Properties of these categories were implicitly obtained already in 1986 through the study of null vector equations [153] for the corresponding conformal blocks. In the early nineties, the tensor structure for theories with negative level was constructed by Kazhdan and Lusztig [151]; this result was transferred to positive level by Finkelberg [79]. It can also be recovered in the more general Huang-Lepowsky theory of tensor products.

Vertex algebras have meanwhile also become an important tool in several areas of pure mathematics, including the algebraic geometry of Hilbert schemes [157] and the geometric Langlands

program (see [86] for a review). Sheaves of vertex algebras also enter in the chiral de Rham complex [167], an attempt for a geometric realization of elliptic cohomology. The locality principle that is integrated in the definition of vertex algebras has also provided a fruitful rationale to single out structures in a wider context of infinite-dimensional algebra [145].

## 3.2 Conformal blocks

An important aspect of vertex algebras is that they give rise to *conformal blocks*, i.e. vector bundles over moduli spaces of curves with marked points. If the vertex algebra is a conformal vertex algebra, then these bundles carry a projectively flat connection, called Knizhnik-Zamolodchikov connection. For vertex algebras based on affine Lie algebras, these vector bundles were first explored by Tsuchiya, Ueno and Yamada [226, 228]. In this case they provide interesting non-abelian generalizations of theta functions and are thus of independent interest in algebraic geometry, see e.g. the Bourbaki talk [217] and [20].

The ranks of these bundles are given by the Verlinde formula; their computation is highly non-trivial. Various different arguments, including complete proofs, have been given for the Verlinde formula for Wess-Zumino-Witten models:

- Heuristic arguments, based on a path integral approach to Chern-Simons theories [49, 28].
- Techniques from algebraic geometry [69, 20]. (For earlier work restricted to  $\mathfrak{sl}(2)$ , see also [27, 224].)
- Fixed points of loop group actions [4].
- Techniques from homological algebra, in particular Lie algebra cohomology, combined with a vanishing argument [222].
- Holomorphic quantization of Chern-Simons theories [13, 143] and related methods from symplectic geometry [170].

For the construction of conformal blocks from vertex algebras we refer to the Bourbaki talk [85] and to the textbook [88]. This construction is also crucial for the relation to the Beilinson-Drinfeld theory of chiral algebras [23]. Conversely, from the values of all  $n$ -point conformal blocks on a suitable finite-dimensional subspace of states, a vertex algebra can be reconstructed [113].

Since one deals with bundles of conformal blocks, and thus multivalued functions instead of single-valued correlators, this theory is not an ordinary physical quantum field theory. Still, it appears directly in applications in physics, such as in the description of universality classes of quantum Hall fluids (for a review see [101]). It should be appreciated that the direction of the magnetic field distinguishes a chirality in a quantum Hall fluid; accordingly the relevance of chiral (as opposed to full local) conformal field theory is rather natural.

There are two major approaches to make conformal blocks explicitly computable.

- Based on a construction of modules over the Virasoro algebra by Feigin and Fuks [71], Dotsenko and Fateev [63] were able to calculate the fusing matrices and bulk structure constants in Virasoro minimal models. The underlying idea [74] is to construct modules over complicated symmetry structures as the BRST cohomology of free field representations. This leads in particular to explicit expressions for solutions of the Knizhnik-Zamolodchikov equations, see e.g. [78]. The BRST approach also brought up several Kazhdan-Lusztig type correspondences [73, 72].

- Another approach uses differential equations, which can be obtained e.g. from null vectors in representations of the chiral algebra. Examples include the Knizhnik-Zamolodchikov equation [153] (for reviews, see [67, 168]) and the Gepner-Witten equation [126]. Zhu introduced [244] a finiteness conditions on vertex algebras, called  $C_2$ -cofiniteness, which does not only guarantee the convergence of the characters of modules over a vertex algebra [244], but also leads to good differential equations for chiral blocks. The same cofiniteness condition implies the existence of Zhu's algebra [90] which constitutes one important tool for the study of chiral conformal field theory.

The differential equations for conformal blocks can be made most explicit for surfaces of low genus. For results in genus one see e.g. [26], and for the particular case of differential equations for characters see e.g. [169].

### 3.3 Modular tensor categories

The categorical structure found in the work of Moore and Seiberg leads to the notion of a modular tensor category. A rigorous construction of three-dimensional topological field theory or, equivalently, of a modular functor, by Reshetikhin and Turaev [196, 197] is based on this notion.

Modular tensor categories also arise naturally in the approach to conformal field theory via the Doplicher-Haag-Roberts framework. Indeed, the category of local sectors of a net of von Neumann algebras on the real line of finite  $\mu$ -index is a (unitary) modular tensor category if the net is strongly additive and has the split property [149].

In the vertex algebra approach to conformal field theory, the relevant result is that the representation category of a self-dual vertex algebra that obeys the  $C_2$ -cofiniteness condition and certain conditions on its homogeneous subspaces is a modular tensor category, provided that this category is semisimple [138]. The result of [138] relies on a careful implementation of the structures unraveled by Moore and Seiberg, combined with Huang-Lepowsky's theory of tensor products of representations of conformal vertex algebras.

The structural importance of the  $C_2$ -cofiniteness condition cannot be overrated. Recent studies of vertex algebras with non-semisimple representation category (see e.g. [172]) indicate that  $C_2$ -cofiniteness suffices, even in the absence of semisimplicity, to endow the representation category with a structure reasonably close to the one of a modular tensor category. This opens the way to a better understanding of logarithmic [131], and possibly also other non-rational, conformal field theories.

Let us make two more comments on the relation between modular tensor categories and vertex algebras:

- It is worth emphasizing that for endowing the representation category of a suitable conformal vertex algebra with the structure of a modular tensor category, only properties of conformal blocks for surfaces of genus 0 and 1 are used. Now from conformal vertex algebras and modular tensor categories one obtains representations of mapping class groups in two different ways. First, via the Knizhnik-Zamolodchikov connection on the bundles of conformal blocks. In this case the mapping class group arises as the fundamental group of the moduli space. And second, via three-dimensional topological field theory and the embedding of mapping class groups in cobordisms. The latter construction uses directly the modular tensor category, while the former is based on the vertex algebra and its conformal blocks. To establish that, for any genus, the two constructions give equivalent representations will require a serious improvement of the understanding of conformal blocks for rational vertex algebras.



- For lattice vertex algebras, the modular tensor category together with the value of the Virasoro central charge is equivalent to the genus of the lattice. The genus of a lattice  $L$  is, by definition, the collection of all local lattices  $L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ , including  $L \otimes_{\mathbb{Z}} \mathbb{R}$ . For a Euclidean lattice  $L$ , the latter is equivalent to the quadratic space  $(L^*/L, q)$  given by the discriminant form and the rank of the lattice. In conformal field theory terms these data correspond to the equivalence class, as a braided monoidal category, of the representation category of the lattice vertex algebra and to the value of the Virasoro central charge.

This suggests [136] to regard the modular tensor category and the Virasoro central charge of a given rational vertex algebra as arithmetic information and raises, in particular, the question of whether good Mass formulas exist for vertex algebras.

The categorical dimensions of simple objects in a modular tensor category are typically not integers. As a consequence, modular tensor categories do not admit fiber functors to the category of complex vector spaces. However, one can show that fiber functors to categories of bimodules over a suitable ring exist. As that ring, one can take the endomorphism ring of any generator of a module category over the modular tensor category. Reconstruction then yields algebraic objects like weak Hopf algebras [29] or other “quantum symmetries” [166, 187]. Conversely, it has been established [182] that connected ribbon factorizable weak Hopf algebras over  $\mathcal{C}$  with a Haar integral have representation categories which are modular tensor categories.

The tensor product functor on a modular tensor category  $\mathcal{C}$  is exact. As a consequence, the Grothendieck group  $K_0(\mathcal{C})$  carries a natural structure of a ring, the so-called fusion ring. It has a natural basis given by the classes of simple objects of  $\mathcal{C}$ . Verlinde’s formula [229] (which for general rational conformal field theories was proven in [138]) relates the structure constants of the fusion ring in this basis to the modular transformations of the characters of vertex algebra modules. This relationship motivated the study of various aspects of fusion rings, which was actively pursued in the early 1990s; for a review see [102].

### 3.4 Classification of modular invariant partition functions

Intrigued by the A-D-E-structure that was found in the classification of modular invariants for minimal models and  $\mathfrak{su}(2)$  WZW models, in the early nineties several groups pursued the program to classify modular invariant torus partition functions. On the one hand, the theory of simple currents, due to Schellekens and Yankielowicz, gave a powerful machinery to construct modular invariant partition functions. (See [207] for a review, and [156] for the general classification.) At that point, exceptional modular invariants, i.e. modular invariants that are not explainable via simple currents and charge conjugation, remained still rather mysterious. On the other hand, Gannon and others were able to obtain remarkable classifications for some particular classes of models; see e.g. [116, 219, 117, 119, 46].

This classification program is put into a different perspective by the observation (see e.g. [206, 230, 106]) that there exist modular invariant bilinear combinations of characters (with non-negative integral coefficients and with unique vacuum) which are unphysical in the sense that they cannot arise as the torus partition function of any consistent local conformal field theory: Finding all such bilinear combinations of characters provides very useful restrictions on the possible form of torus partition functions, but is not equivalent to classifying consistent conformal field theories. (The existence of spurious solutions should not come as a surprise, since many more conditions are to be satisfied: the sewing constraints, which implement compatibility of the correlators under cutting and gluing of surfaces, see e.g. [216, 160, 186].) Strikingly, the true diagonal modular

invariant is not necessarily physical (for a counter example see [218]); a classification of those modular tensor categories for which the diagonal modular invariant is physical is still unknown.

### 3.5 Sigma model approaches to conformal field theory

The motivation to consider sigma model approaches, i.e. models based on spaces of maps  $\Sigma \rightarrow M$ , include extending WZW models to noncompact groups [121] as well as understanding cosmologically interesting backgrounds in string theory [54]. Sigma models have provided strong links between quantum field theory and various aspects of geometry, in particular complex and symplectic geometry, leading [135] to common generalizations of both.

There is a vast literature on conformal sigma models, which we are unable to review appropriately. We restrict our attention to developments that have direct relation to *rational* conformal field theory. Early important work by Felder, Gawędzki and Kupiainen [77] obtains in a path integral approach the quantization of the level and the bulk spectra of Wess-Zumino-Witten sigma models on compact groups that are not necessarily simply connected. For an extension to coset conformal field theories, we refer to [123, 147].

String vacua based on rational conformal field theories were first constructed by Gepner [124]; his construction was later generalized by Kazama and Suzuki [150]. Remarkably, many of these models can be matched with string compactifications on Calabi-Yau manifolds. The symmetry induced by conjugation of the  $U(1)$  charge in a rational  $N=2$  superconformal field theory is in this way at the basis of mirror symmetry of Calabi-Yau spaces [159].

### 3.6 Topological field theory from path integrals and state sums

Witten's work on Chern-Simons theories has been extended in various ways. Because of the close relation between chiral conformal field theory and three-dimensional topological field theory, we mention here some aspects of this development. Topological field theories based on finite groups were considered by Dijkgraaf and Witten [55]; the paper presents structural relations to WZW theories based on non-simply connected compact Lie groups.

Lagrangian descriptions of topological field theories have been influential in several directions. For instance, Chern-Simons perturbation theory gave rise to Vassiliev invariants (for a review see [231]). The theory of Vassiliev invariants has found applications in other fields of mathematics as well, e.g. to universal Lie algebras [232] and to holomorphically symplectic manifolds in the form of Rozanski-Witten [198] invariants.

Another construction of topological field theories is via state sum models. These have been discussed in various dimensions, including lattice topological field theories in two [110] and three [43] dimensions, as well as the Turaev-Viro construction [227]. The latter implements an old idea of Ponzano and Regge [185] to build invariants with the help of  $6j$ -symbols for tensor categories and yields, in the case of modular tensor categories, an invariant of three-manifolds that is the absolute square of the Reshetikhin-Turaev invariant.

## 4 New frontiers and categorical structures

Dual models started out as theories describing hadrons as bound states of charged particles which were supposed to be located at the end points of an open string. As a consequence, the underlying surfaces on which the theory is considered were allowed to have non-empty boundary. In contrast, the first two periods of the development of rational conformal field theory covered in this review

were largely concerned only with closed orientable surfaces. But in order to study defects in systems of condensed matter physics [188], percolation probabilities [39], (open) string perturbation theory in the background of the string solitons known as D-branes [191], and order-disorder dualities, it became again necessary to consider also surfaces that may have boundaries and / or can be non-orientable, as well as surfaces with defect lines.

More specifically, the advent of D-branes in string theory brought boundary conditions in rational conformal field theories to the center of interest. Based on earlier work of Cardy [38] and of the Rome group (in particular [186], for a review see [10]), it became evident that boundary conditions provide crucial new *structural* insight into conformal field theories. In particular, the fundamental importance of the two-fold oriented cover  $\widehat{X}$  of  $X$  (which had already been noted in the study of dual models, see section 2 above) gained again fundamental importance: full local conformal field theory on a conformal surface  $X$  is related to a chiral conformal field theory on  $\widehat{X}$ .

## 4.1 Structure constants

Early progress in the study of conformal field theory on surfaces with boundary focussed on the computation of structure constants for operator products. Similarly as in the case of closed surfaces, for which structure constants had been obtained (see e.g. [63, 243, 42, 104, 190]) by analyzing four-point correlators of bulk fields on the sphere, the first complete results for the structure constants on surfaces with boundaries were obtained for Virasoro minimal models [200, 201].

A strong focus was on the computation of one-point correlators of bulk fields on a disk, which may be collected in the coefficients of so-called boundary states [142]. They provide significant information of much interest for applications, like ground state degeneracies [2] or Ramond-Ramond charges of string compactifications [33]. Moreover, they encode the integral coefficients of the annulus partition functions and thus give the spectrum of boundary fields. Ramond-Ramond charges are related to twisted K-theory [171, 239]. This leads to the interpretation of the Verlinde algebra of Wess-Zumino-Witten models as equivariant twisted K-theory of compact, connected and simply-connected Lie groups (see [83] for a review), and to conformal field theory techniques for computing D-brane charges [82, 32].

In string compactifications based on a rational conformal field theories, such as those obtained by Gepner's [124] construction, the boundary states of the conformal field theory correspond to specific D-branes in the string theory (see e.g. [33]). The construction of boundary states for such theories turns out to be remarkably subtle. In the case of Gepner models, it was first discussed in [194]. For the complete construction, including a correct treatment of field identification fixed points, see [108] and [103].

An important structural observation [21] concerns the annulus coefficients: they furnish matrix-valued representations with non-negative integer entries (so called NIMreps) of the fusion ring. Even more important was the insight [22, 75] that the constraints on structure constants for boundary fields preserving a given boundary condition give rise to generalized pentagon relations. The mixed  $6j$ -symbols appearing in these relations have the conceptual interpretation as multiplication morphisms of an associative algebra in the modular tensor category of chiral data. This algebra has the physical interpretation as the algebra of boundary fields for a fixed boundary condition.

## 4.2 TFT, Frobenius algebras and CFT correlators

In the rational case the algebra of boundary fields turns out to have the mathematical structure of a special symmetric Frobenius algebra in the modular tensor category of chiral data. This Frobenius algebra is the crucial ingredient in a construction that describes the correlation functions of rational conformal field theories (RCFTs) in terms of invariants provided by three-dimensional topological field theories (TFTs); for a review see [211]. Special symmetric Frobenius algebras also appear in the approach to conformal field theory via nets of von Neumann algebras, where they are realized [163, 68, 164] as so-called  $Q$ -systems [162].

In the description via TFT, the RCFT correlators are realized as invariants of ribbon graphs in three-manifolds under the Reshetikhin-Turaev modular functor. Thereby geometric structure in three dimensions is again used to gain insight into two-dimensional theories, in this case into full *local* conformal field theories. The relevant three-manifold geometry also appears in the study of CFT on world sheet orbifolds with the help of Chern-Simons theory [137].

The description of full local CFT via TFT allows one in particular to recover modular invariant partition functions. The problem (which has spurious unphysical solutions) of classifying bilinear combinations of characters as candidates for modular invariant partition functions is thereby replaced [105] by the problem of classifying Morita classes of special symmetric Frobenius algebras or, equivalently [189], appropriate module categories. Moreover, the formalism treats exceptional modular invariants and simple current invariants on the same footing. For instance, for the tensor categories based on the  $\mathfrak{sl}(2, \mathbb{C})$  Lie algebra, the Cappelli-Itzykson-Zuber A-D-E classification is recovered [152].

The bulk theory is obtained from the Frobenius algebra of the boundary theory as a commutative special symmetric Frobenius algebra in the product of the modular tensor category of the RCFT with its opposed category. This relation between boundary and bulk generalizes the situation found [174] in two-dimensional topological field theories, where both the boundary Frobenius algebra and the bulk Frobenius algebra are algebras in the category of complex vector spaces. This structure can also be interpreted in terms of vertex algebras, leading to so-called full field algebras [139] and open-closed field algebras [154].

The TFT construction provides a detailed dictionary between algebraic notions and physical concepts. A generalization of an algebra with involution, a so-called Jandl structure on  $A$ , allows one to construct also correlators for unoriented surfaces. Semisimple Frobenius algebras with an involution had appeared before in the study of two-dimensional lattice topological field theories [148]. Symmetries and order-disorder dualities of RCFT correlators can be conveniently described with the help of defect lines [99], which in turn have a natural representation within the TFT formulation of the CFT correlators.

## 4.3 Some current lines of research

In the last couple of years, new and important lines of research in conformal field theory have come to the foreground. Here, we give a choice that is, of course, again biased.

### 4.3.1 Beyond semisimple rational conformal field theories

At present, many efforts are devoted to a better understanding of conformal field theories based on chiral symmetry structures whose representation categories are not semisimple any longer.

The most important example is logarithmic conformal field theory, a generalization of rational conformal field theory dating back to the early nineties [131]. Much insight has been gained for

the particular class of  $(1, p)$ -models for which a Kazhdan-Lusztig correspondence to a Hopf algebra exists (see [115] for a recent summary). Logarithmic conformal field theories are also important for their close relation to models of statistical mechanics, e.g. to percolation models [193].

Other important subclasses of theories are non-compact theories like Liouville theory and its relatives (for a review, see [223]) and sigma models with supersymmetric target spaces [202] which currently find applications both in the context of the AdS/CFT duality [3] and in statistical mechanics, e.g. in the description of transitions between quantum Hall plateaux [245].

### 4.3.2 Higher categorical geometry for target spaces

In Lagrangian approaches to conformal field theories, higher categorical structures have become more and more important as well. Already in the mid eighties, Alvarez [7] and Gawędzki [120] noticed that the Wess-Zumino term is closely related to Deligne hypercohomology. Brylinski [34], Murray [179] and others then constructed hermitian bundle gerbes with connection as a geometric realization of hypercohomology. This allows in particular for a geometric understanding of the Wess-Zumino term as a surface holonomy for these gerbes. This description was extended to D-branes by the notion of gerbe modules [122] including non-abelian effects, to holonomy for unoriented surfaces with the introduction of Jandl structures for gerbes [209], and to a target space interpretation of defect lines with the notion of bibranes [109].

One current point of interest are extensions to topological field theories that also associate values to 1- and 2-manifolds. They require higher category theory, see e.g. [84] for a recent discussion based on compact Lie groups. An early contribution to this area was published in this journal [14]. Higher categories and higher-dimensional algebra have become an increasingly important topic of research, including the idea [47] to use categorification to extended techniques from three-dimensional topological field theory to higher dimensions.

Another area in which category theory and conformal field theory meet is the study of D-brane categories. To review their role in homological mirror symmetry [155] is beyond the scope of our survey; for an exposition that emphasizes conformal field theory aspects in terms of sigma models, we refer to [11].

The relation between three-dimensional topological and two-dimensional conformal field theory leads to a corresponding relation between higher categorical structures: the hermitian bundle gerbe with connection on a compact connected group  $G$  describing the Wess-Zumino term of the two-dimensional theory is a 2-categorical structure. If this gerbe admits a multiplicative structure, it can be lifted [41] to a 2-gerbe on the classifying space of  $G$ , and thus to a 3-categorical structure. The three-holonomy of a corresponding 2-gerbe with connection describes the Chern-Simons term of the associated three-dimensional topological field theory.

### 4.3.3 SLE

Another field of mathematics that shed a new light on conformal field theory are stochastic differential equations. Stochastic Loewner Evolution (SLE) was defined by Schramm [208]; for reviews see e.g. [234, 40]. SLE (or, more precisely, chordal SLE) provides a measure on curves on the upper half plane which start at zero and end at infinity, as well as on domains conformally equivalent to the upper half plane. The measure satisfies three conditions: conformal invariance, reflection symmetry, and a restriction property. These properties turn out to fix the measure up to a real parameter  $\kappa \geq 0$ , which is related to the Virasoro central charge by  $c = (3\kappa - 8)(6 - \kappa)/(2\kappa)$ .

In certain examples it has been established that the SLE measure on curves arises as a continuum limit of the weight of spin configurations with a prescribed domain boundary in two-

dimensional lattice models. One such example is critical percolation, where crossing formulas conjectured by Cardy [39] and by Watts [233] using boundary conformal field theory were proved in [214] and [66], respectively.

The factorization of amplitudes is expected from the heuristic path integral formulation of CFT. The approaches to CFT reported in sections 4.1 and 4.2 provide a mathematical formalization of this property. Approaches based on SLE, on the other hand, directly provide a microscopic description by replacing the path integral by a precisely defined measure. This provides a much more direct relation to lattice models, but it is by no means straightforward to express CFT correlators in the SLE formulation. For example, conjecturally, correlators with insertions of the stress-energy tensor can be obtained from the probability that the SLE curve passes through a number of small intervals centered at the insertion points [65] (this is at central charge  $c=0$ ; for  $c$  between 0 and 1 see [64]), and there is a relation between Virasoro null vectors and SLE martingales [19].

## 5 Omissions

There are important aspects of two-dimensional rational conformal field theories that we could not cover in this review. Some of them are the following:

- Applications of conformal field theory to entanglement entropy [35].
- Applications of conformal field theory to quantum computing [181].
- D-branes in WZW models, in particular their target space geometry, [5, 75].
- Renormalization group flows between conformal field theories, including Zamolodchikov’s  $c$ -theorem [241] and an analogous result for flows between conformal boundary conditions [2, 91].
- Integrable [242, 127] and numerical [240, 62] methods for investigating bulk and boundary perturbations of conformal field theories.
- The generalization of the fermion-boson correspondence to higher genus world sheets (see e.g. [9] for an early discussion), and applications of the fermion-boson correspondence in combinatorics, see e.g. [25].
- Galois symmetries acting on chiral data [50], and their use in the study of modular invariants [45, 106] and in the proof of the congruence subgroup conjecture [17].
- The relation between fusion rings and quantum cohomology, see e.g. [125, 238].
- Applications of conformal and superconformal field theory in elliptic cohomology [212, 220].

That the topics in this list, and more, have not even been treated briefly in this survey illustrates what an enormous amount of knowledge about rational conformal field theory has been compiled during the last quarter century.

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