OPEN STRINGS AND
3D TOPOLOGICAL FIELD THEORY

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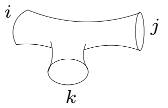
In the perturbative formulation of string theory the partition function in the background of D-branes can be expanded as

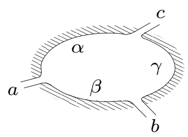
$$Z_{\text{string}} = \sum_{g,b} g_s^{-2+2g+b} Z_{g,b} \,. \tag{1}$$

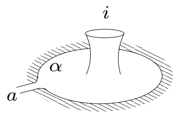
Here $Z_{g,b}$ is the value of the diagram given by a Riemann surface of genus g with b boundary components. One way to compute the numbers $Z_{g,b}$ is to work in conformal gauge. In this gauge one obtains a conformal field theory (CFT) living on the string world sheet, and the $Z_{g,b}$ result from integrating correlators of this CFT over the moduli space of the Riemann surface. In order for this approach to make sense, the CFT has to be well defined on surfaces of arbitrary genus and with an arbitrary number of boundaries. For type I strings also non-orientable surfaces have to be included in the expansion (1). In this note we restrict ourselves to type II strings and orientable world sheets.

With this motivation in mind we set ourselves the aim to construct a CFT consistent on all surfaces relevant in (1). This can be treated entirely as a problem in CFT. We will not worry about whether a particular CFT actually appears in the conformal gauge description of some string background.

One approach to obtain CFT correlators on all surfaces is via sewing. Denote the space of closed string states (i.e. the space of states associated to a circle) by $\mathcal{H}^{\mathrm{closed}}$ and the space of open string states (associated to an interval) by $\mathcal{H}^{\mathrm{open}}_{\alpha,\beta}$. Here α, β belong to a set $\mathcal{B}$ of boundary conditions we want to allow at either end of the open string. By repeatedly cutting along circles and intervals one can decompose every orientable world sheet into the following building blocks:

C_{ijk} : 

$c^{\alpha\beta\gamma}_{abc}$: 

B^{α}_{ai} : 

(2)

Here $i, j, k \in \mathcal{H}^{\mathrm{closed}}$, α, β, γ are boundary conditions and a, b, c, d are open string states. By conformal invariance the building blocks (2) can be mapped to the correlators of three bulk fields on the sphere, three boundary fields on the disc, and one bulk plus one boundary field on the disc, respectively. Again by conformal invariance, these are determined by three sets of constants C_{ijk}, $c^{\alpha\beta\gamma}_{abc}$ and B^{α}_{ia}. From these constants together with the factors arising from the conformal transformation one can obtain the correlator on the original world sheet X by summing over all intermediate states on the cuts.

Conversely, one can ask the question when a given a set of data

$$\mathcal{H}^{\mathrm{closed}}\ ,\quad \mathcal{B}\ ,\quad \mathcal{H}^{\mathrm{open}}_{\alpha,\beta}\ ,\quad C_{ijk}\ ,\quad c^{\alpha\beta\gamma}_{abc} \quad \text{and} \quad B^{\alpha}_{ia} \tag{3}$$

leads to a CFT that is consistent on all surfaces. For this to be the case the correlator on X obtained from sewing the building blocks (2) must be independent of the way one has chosen to cut up the world sheet X. This leads to an infinite set of nonlinear constraints on the infinite set of constants (3).

In a rational CFT one can obtain a finite system of equations. In such theories the symmetry algebra $\mathcal{A}$ is large enough to decompose the state spaces $\mathcal{H}^{\mathrm{closed}}$ and $\mathcal{H}^{\mathrm{open}}_{\alpha,\beta}$ into finitely many irreducible representations. The nonlinear constraints can then be formulated in terms of a finite subset of the constants (3), namely those which only involve primary fields. Constants involving non-primary fields are related to the former by $\mathcal{A}$. In this way one can obtain a finite, sufficient and necessary set of polynomial relations for the data (3), the sewing constraints [1, 2].

One of the sewing constraints is the well known requirement that the torus partition function has to be modular invariant. To construct a consistent CFT via the method of sewing it is however necessary to work out the full set of sewing constraints, which in general is a difficult problem since they form an overdetermined system of non-linear equations.

From this point of view it would be nice to have a simple criterion to ensure that a set (3) of data solves the sewing constraints. One of the results in [3, 4] is that in order to construct a CFT consistent on all orientable Riemann surfaces, it is enough to find consistent amplitudes of boundary fields on a disc:

> Select a rational chiral algebra $\mathcal{A}$. If one can find constants $c^{\alpha\alpha\alpha}_{abc}$ for one boundary condition α preserving $\mathcal{A}$ such that the correlator of four boundary fields on the disc is consistent with the two different ways of sewing, then one can construct all the remaining data in (3), where $\mathcal{B}$ will be the set of all elementary boundary conditions preserving $\mathcal{A}$.

There are several surprising points about this result. First, it is possible to construct a consistent *bulk* theory starting only from the constants $c^{\alpha\alpha\alpha}_{abc}$ fixing the 3-point functions of *boundary* fields. Second, the consistency requirement on the $c^{\alpha\alpha\alpha}_{abc}$ is a constraint arising from a correlator on a disc, i.e. a constraint at genus zero. Still the CFT constructed by the method in [3, 4] is consistent on surfaces of arbitrary genus. Third, starting from a single boundary condition preserving the chiral algebra $\mathcal{A}$ one obtains all other boundary conditions with this property. As an input for the construction of the CFT one needs to work out the representation theory of $\mathcal{A}$ (a hard problem in itself), as well as to give the constants $c^{\alpha\alpha\alpha}_{abc}$. But once this has been achieved, the computation of the data (3) is reduced to solving linear problems.

Below we give a brief sketch of how to obtain the above result. The proof makes extensive use of results of [5, 6]. It has four ingredients, which we will present for the special case of WZW models.

(i) The correlator on a world sheet X is an element of the space of conformal blocks $\mathcal{H}(\hat{X})$ on the complex double $\hat{X}$ of X. Conformal blocks are multivalued analytic functions which are obtained as solutions to the Knizhnik-Zamolodchikov equations. The double $\hat{X}$ of a surface X is a double cover of X with the two sheets identified along the boundary of X. For example for $X = S^2$ one has $\hat{X} = S^2 \sqcup S^2$, and for X a disc one finds $\hat{X} = S^2$.

(ii) In Chern-Simons theory there is a space of states associated to each two-dimensional boundary of the original three-manifold. It turns out that this space of states can be identified with the vector space $\mathcal{H}(\hat{X})$ of conformal blocks on the surface $\hat{X}$ [7].

(iii) To describe an element in the space $\mathcal{H}(\hat{X})$ one can use the Chern-Simons path integral. One considers a three-manifold M with embedded Wilson graph so that the boundary ∂M is $\hat{X}$. The Wilson lines are allowed to end on the boundary, at the insertion points of chiral fields for the conformal blocks.

(iv) Given a surface X, possibly with field insertions and boundaries, there is a systematic construction of a three-manifold M_X (the connecting manifold) and a Wilson graph in M_X such that $\partial M_X = \hat{X}$ and such that the conformal block in $\mathcal{H}(\hat{X})$ described by the Chern-Simons path integral for M_X is precisely the correlator on X [6, 3].

One may wonder where the constants $c_{abc}^{\alpha\alpha\alpha}$ enter in this procedure. They are needed in the construction of the Wilson graph in (iv) as they determine the representation labels attached to some of the Wilson lines as well as the intertwiners to be used at points where three of those Wilson lines meet.

In fact we do not need to restrict ourselves to WZW models and Chern-Simons theory. Instead one can use directly the functorial definition of a three-dimensional topological field theory. The idea is to use (the conjecture) that the representation category of a rational vertex algebra $\mathcal{A}$ is a modular category. To each modular category one can assign a 3d TFT which is defined as a functor from a cobordism category to the category of vector spaces [8]. In this way one avoids having to think about path integrals and actions, and the objects one uses are mathematically well defined and convenient for computations.

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