Conformal field theory and algebra in braided tensor categories II

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In part I of this series of two talks we have seen that the data decorating a conformal field theory world sheet is taken from the bicategory $\mathcal{F}rob(\mathcal{C})$ of special symmetric Frobenius algebras in the modular tensor category $\mathcal{C}$ provided by the representations of a suitable vertex operator algebra $\mathcal{V}$.

A Frobenius algebra $A \in \mathcal{F}rob(\mathcal{C})$ labels a full CFT with chiral symmetry $\mathcal{V}$. Part of the data of such a CFT is the space of bulk fields B. This is an object in the product category $\mathcal{C}_+ \boxtimes \mathcal{C}_-$, the $\mathbb{C}$-linear category whose objects are direct sums of pairs of objects, and whose morphism spaces are tensor products (over $\mathbb{C}$) of those in $\mathcal{C}$. The signs $\pm$ refer to the braiding and twist, $\mathcal{C}_+$ is just equal to $\mathcal{C}$, and $\mathcal{C}_-$ is $\mathcal{C}$ with inverse braiding and twist. The space of bulk fields B is associated to a marked point in the interior of the world sheet, and by evaluating correlators with several marked points on a sphere, the object B gets equipped with the structure of a commutative symmetric Frobenius algebra. We will now give a direct construction of B, starting from the modular tensor category $\mathcal{C}$ and a special symmetric Frobenius algebra $A \in \mathcal{C}$.

The starting point is a functor $R : \mathcal{C} \to \mathcal{C}_+ \boxtimes \mathcal{C}_-$ which is left and right adjoint to the tensor product functor T. The functor R acts on objects as

$$R(V) = \bigoplus_{i \in \mathcal{I}} (V \otimes U_i^\vee) \boxtimes U_i \ ,$$

where $\mathcal{I}$ is a set of labels for representatives U_i of the isomorphism classes of simple objects in $\mathcal{C}$ and $(\ )^\vee$ denotes the dual. One checks that R is naturally isomorphic to $R'(V) = \bigoplus_{i \in \mathcal{I}} U_i^\vee \boxtimes (V \otimes U_i)$. We will use R it what follows. It is in general not a tensor functor (it is so only if $\mathcal{C}$ is equivalent to the category of finite dimensional complex vector spaces), but it is a lax and colax tensor functor, and one can show that if A is a special symmetric Frobenius algebra, so is $R(A)$ [1, Prop. 2.25]. Typically, $R(A)$ is not commutative.

The algebra B we aim to describe is commutative, and turns out to be the centre of $R(A)$. To be more precise, in a braided tensor category there are two notions of a centre: The *left centre* $C_l(A)$ of an algebra A is the maximal subobject of A such that $m \circ c_{A,A} \circ (e \otimes id_A) = m \circ (e \otimes id_A)$, where m is the multiplication morphism of A, c is the braiding of $\mathcal{C}$, and e is the subobject embedding; the *right centre* $C_r(A)$ is the maximal subobject of A such that $m \circ c_{A,A} \circ (id_A \otimes e) = m \circ (id_A \otimes e)$. The left and right centre are in general not isomorphic, and possibly not even Morita equivalent. However, one can single out a preferred class of modules, the so-called local modules, and show that the categories of local $C_l(A)$-modules and local $C_r(A)$-modules are equivalent [2, Thm. 5.20].

The space of bulk fields of a CFT motivates a third notion of a centre.

Definition 1. [3, Def. 4.9] *The* full centre *of a special symmetric Frobenius algebra A in a modular tensor category $\mathcal{C}$ is $Z(A) = C_l(R(A)) \in \mathcal{C}_+ \boxtimes \mathcal{C}_-$.*

That C_l appears instead of C_r is linked to a choice made when defining the lax and colax tensor structure on R (cf. [1]). The full centre $Z(A)$ is a commutative special symmetric Frobenius algebra [3, Lem. 4.10], which contains $C_l(A) \boxtimes \mathbf{1}$ and $\mathbf{1} \boxtimes C_r(A)$ as subalgebras. It has another interesting property which neither the left nor right centre provide: $Z(A)$ separates Morita classes.

Theorem 1. [4, Thm. 1.1] *Let $\mathcal{C}$ be a modular tensor category and let A, B be simple special symmetric Frobenius algebras in $\mathcal{C}$. Then $Z(A) \cong Z(B)$ as algebras if and only if A and B are Morita equivalent.*

In CFT, the interpretation of $Z(A)$ is that as an object in $\mathcal{C}_+ \boxtimes \mathcal{C}_-$ it describes the space of bulk fields associated to a marked point in the interior of the world sheet, and its counit and multiplication encode the correlator of a sphere with three marked points. One can ask if every commutative symmetric Frobenius algebra $B \in \mathcal{C}_+ \boxtimes \mathcal{C}_-$ can be written in the form $Z(A)$ for some $A \in \mathcal{F}rob(\mathcal{C})$. This turns out to be true if we impose two more conditions: the algebra B must be simple and its quantum dimension must coincide with the global dimension of $\mathcal{C}$. In CFT these two conditions refer to the uniqueness of the bulk vacuum and modular invariance of correlators on the torus.

Theorem 2. [1, Thm. 3.4, 3.22] *Let $\mathcal{C}$ be a modular tensor category and let B be a simple commutative symmetric Frobenius algebra in $\mathcal{C}_+ \boxtimes \mathcal{C}_-$ with $\dim(B) = \mathrm{Dim}(\mathcal{C})$. Then there exists a simple special symmetric Frobenius algebra $A \in \mathcal{C}$ such that $B \cong Z(A)$ as Frobenius algebras.*

This means that every CFT which is defined on genus zero closed oriented surfaces, has the same rational vertex operator algebra $\mathcal{V}$ as left and right moving chiral symmetry, has a unique bulk vacuum, and is modular invariant on the torus, can be extended to a consistent set of correlators on open/closed world sheets.

If we describe a full CFT by an object $A \in \mathcal{F}rob(\mathcal{C})$, we have automatically also singled out a preferred boundary condition, namely the one labelled by A. Equivalent CFTs with different preferred boundary conditions correspond to Morita equivalent algebras in $\mathcal{C}$. In view of this it would be desirable to have a Morita invariant formulation of the datum defining a CFT. This is provided by the notion of module categories, which can be understood as the categorification of a module over a ring: A *right module category* over a tensor category $\mathcal{C}$ is a category $\mathcal{M}$ together with a bifunctor $\odot : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ and associativity and unit isomorphism subject to coherence conditions, see, e.g., [5].

Given an algebra $A \in \mathcal{C}$, the category A-mod of left A-modules is a module category over $\mathcal{C}$ via $M \times V \mapsto M \otimes V$. Under suitable assumptions, in particular semi-simplicity and finiteness of the categories $\mathcal{C}$ and $\mathcal{M}$, there is a converse statement [5, Thm. 1]: The *internal end* of an object $M \in \mathcal{M}$ is the object $\underline{\mathrm{End}}(M) \in \mathcal{C}$ representing the functor $V \mapsto \mathrm{Hom}(M \odot V, M)$. It comes quipped with the structure of an algebra, and $\underline{\mathrm{End}}(M)$-mod is equivalent, as a module category, to $\mathcal{M}$.

The CFT interpretation of a module category $\mathcal{M}$ over the modular tensor category $\mathcal{C}$ is that the objects of $\mathcal{M}$ are the boundary conditions compatible with the chiral symmetry, and the internal end $\underline{\mathrm{End}}(M)$ for a given $M \in \mathcal{M}$ is the space of boundary fields assigned to a marked point on a boundary segment labelled by M. In fact, we can replace the bicategory $\mathcal{F}rob(\mathcal{C})$ by an equivalent bicategory whose objects are (suitable) module categories over $\mathcal{C}$, and whose morphisms from $\mathcal{M}$ to $\mathcal{N}$ are given by the category of module category functors $\mathcal{F}un(\mathcal{M},\mathcal{N})$.

We will now see how to construct also the commutative Frobenius algebra B associated to marked points in the bulk directly from the module category. Given a module category $\mathcal{M}$ over $\mathcal{C}$ we define two functors α^+ and α^- from $\mathcal{C}$ to $\mathcal{F}un(\mathcal{M},\mathcal{M})$, called *braided induction*. On objects they act in the same way, $\alpha^{\pm}_V$ is the functor $M \mapsto M \odot V$. One also has to provide an isomorpism $\alpha^{\pm}_V(M \odot U) \xrightarrow{\sim} \alpha^{\pm}_V(M) \odot U$. This is done with the braiding of $\mathcal{C}$, and the sign $\pm$ tells us to take either the braiding or its inverse.

We turn $\mathcal{F}un(\mathcal{M},\mathcal{M})$ into a module category over $\mathcal{C} \boxtimes \mathcal{C}$ via a bifunctor $\circledast$: $\mathcal{F}un(\mathcal{M},\mathcal{M}) \times \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{F}un(\mathcal{M},\mathcal{M})$, given on objects by $F \circledast (U \boxtimes V) = \alpha^+_U \circ F \circ \alpha^-_V$. Let $Id_{\mathcal{M}}$ be the identity functor on $\mathcal{M}$ and define

$$Z_{\mathcal{M}} = \underline{\mathrm{End}}(Id_{\mathcal{M}}) \ \in \mathcal{C} \boxtimes \mathcal{C} \ .$$

Theorem 3. *Let $\mathcal{C}$ be a modular tensor category and let A be a special symmetric Frobenius algebra in $\mathcal{C}$. Then $Z_{A-\mathrm{mod}} \cong Z(A)$ as algebras.*

A corresponding statement holds for an algebra A over a field k: the endomorphisms of the identity functor on A-mod are isomorphic, as an algebra, to the centre of A.

In CFT terms, this theorem tells us that the internal end of the identity functor on the category of boundary conditions $\mathcal{M}$ provides us with the space of bulk fields. More generally, for $F \in \mathcal{F}un(\mathcal{M},\mathcal{M})$, the internal end $\underline{\mathrm{End}}(F)$ describes the space of defect fields, i.e. the space of fields associated to a marked point on the world sheet that lies on a defect line labelled by F. The bulk fields occur as the special case of defect fields on the invisible defect, labelled by the identity functor.

References

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