Kramers-Wannier duality from conformal defects

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We demonstrate that the fusion algebra of conformal defects of a two-dimensional conformal field theory contains information about the internal symmetries of the theory and allows one to read off generalisations of Kramers-Wannier duality. We illustrate the general mechanism in the examples of the Ising model and the three-states Potts model.

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Kramers and Wannier found a high/low temperature duality for the Ising model \( \mathbb{I} \) that asserts that a correlator of Ising spins \( \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle \) at inverse temperature \( \beta \) is equal to a disorder correlation function \( \langle \mu_{x_1} \cdots \mu_{x_n} \rangle \) at the dual inverse temperature \( \tilde{\beta} = -\frac{1}{2} \ln \tanh \beta \). In the disorder correlator, the couplings between neighbouring spins dual to the links of \( n/2 \) lines, with each of the positions \( x_k \) at the end of one of the lines, are chosen to be antiferromagnetic (opposite to the standard ferromagnetic nearest-neighbour coupling). This duality has since been considerably generalised, see e.g. \[2, 3\].

The significance of Kramers-Wannier duality lies in the fact that it relates the high-temperature expansion (weak coupling regime) of a lattice model to its low-temperature expansion (strong coupling regime) and thereby makes the latter accessible to perturbation theory.

Kramers-Wannier-like dualities are also a useful tool in understanding the phase structure of a lattice model. At zero magnetic field, the Ising model has a critical point when \( \beta = \tilde{\beta} \). Its universality class is described by a two-dimensional conformal field theory (CFT) with central charge \( c = \frac{1}{2} \). Physical quantities like critical exponents can then be determined by a CFT calculation, relating them to scaling dimensions of bulk fields. The critical Ising model is self-dual under Kramers-Wannier duality, so that a correlator involving spin and disorder fields is equal to another correlator in the same CFT, but with spin fields and disorder fields interchanged.

It is clearly desirable to be able to read off the possible high/low temperature dualities leaving a given critical model fixed solely from knowing its universality class, i.e., its CFT description. In this letter, we provide such a method by relating order/disorder dualities of CFT correlators to conformal defects. Not every defect can be used to establish a duality, but only what we will call ‘duality defects’. Below we present a method that allows us to identify such defects by studying the fusion algebra of all conformal defects. Duality defects relate perturbations of a CFT in different marginal directions, thus allowing one to explore the vicinity of a model in its moduli space, and they also relate different relevant directions, allowing one to extend the order/disorder duality of the CFT to a genuine high/low temperature duality away from the critical point.

Defects in the critical Ising model

Before exhibiting the underlying mechanism in generality, we investigate in some detail the critical Ising model as a first non-trivial example. At central charge \( c = \frac{1}{2} \) the Virasoro algebra has three unitary irreducible highest-weight representations, which we denote by \( \mathbf{1}, \sigma, \varepsilon \). Their weights are \( h_1 = 0, h_\sigma = \frac{1}{16} \) and \( h_\varepsilon = \frac{1}{2} \). Correspondingly, there are three primary bulk fields, the identity \( \mathbf{1} \), the spin field \( \sigma(z) \) and the energy field \( \varepsilon(z) \), with chiral/antichiral conformal weights \((0,0), (\frac{1}{16}, \frac{1}{16})\) and \((\frac{1}{2}, \frac{1}{2})\), respectively.

Next, we introduce conformal defects. One can think of a conformal defect on a surface as being obtained by cutting the surface along the defect line and re-joining the two sides of the cut by an appropriate boundary condition, i.e. a prescription on how bulk fields behave when crossing the cut. This prescription must preserve the conformal symmetry, i.e. both the chiral and antichiral components \( T(z) \) and \( \bar{T}(\bar{z}) \) of the conformal stress tensor must vary continuously across the cut. In contrast, other bulk fields are permitted to exhibit a more complicated behaviour. In fact, dragging a conformal defect across a bulk field other than the stress tensor generally results in disorder fields, as illustrated in figure \[1\].

Because the defect line commutes with the stress tensor, it can be continuously deformed without changing the value of a correlator. In this sense a conformal defect is tensionless. Defect lines can only start and end on field insertions. Such fields are called disorder fields. Since a defect is invisible to \( T \) and \( \bar{T} \), disorder fields fall into representations of two copies of the Virasoro algebra, just as the bulk fields do.

By an argument similar to one used in the analysis of conformal boundary conditions \[4\], in the Ising model one finds three conformal defects \[5\]. They are labelled by the three \( c = \frac{1}{4} \) irreps of the Virasoro algebra. The de-
The effect of pulling a defect past a spin field. Collapsing the circular $\sigma$-Wilson line on the rhs generates the TFT-representation of the disorder field $\mu(z)$.

FIG. 3: Taking defects of type $\sigma$ and $\varepsilon$ past field insertions. The TFT-representation of a) is given in figure 2.
FIG. 4: Order/Disorder duality of a correlator of four spin fields on a sphere, and of two spin fields on a torus, as induced by the $\sigma$-defect.

boundaries. In the Ising model, the boundary conditions are again labelled by the $c = \frac{1}{2}$ irreps [5]: 1 and $\epsilon$ describe fixed boundary conditions with ‘spin up’ and ‘spin down’, respectively, while $\sigma$ describes the ‘free’ boundary condition. Owing to the Ising fusion rules $\sigma \otimes 1 = \sigma \otimes \epsilon = \sigma$ and $\sigma \otimes \sigma = 1 + \epsilon$, a $\sigma$-defect in front of a ‘spin up’ or a ‘spin down’ boundary condition can be replaced by a ‘free’ boundary condition without defect, while a $\sigma$-defect in front of a ‘free’ boundary condition yields the sum of a ‘spin up’ and a ‘spin down’ boundary condition. One thus obtains the well-known duality of fixed and free boundary conditions [3].

So far, we have considered the order/disorder duality only at the critical point. However, the rules listed in figure 3 also allow us to establish the duality away from the critical point. For example, note that taking a $\sigma$-defect through the energy field $\varepsilon(z)$ results in a change of sign. Perturbing the CFT by $\varepsilon(z)$ amounts to a change of temperature, and applying the duality to each term in a perturbation series leads to the equality

$$\langle \sigma(x) \sigma(x') e^{-\lambda f_\varepsilon(y) d^2 y} \rangle = \langle \mu(x) \mu(x') e^{\lambda f_\varepsilon(y) d^2 y} \rangle$$

for the example of a two-point correlator on the sphere.

The general mechanism

We are now in a position to describe a general mechanism that works for all unitary rational conformal field theories. For such models, there is a finite set of primary bulk fields $\phi_\alpha(z)$. One denotes the number of such fields transforming in representations $i$ and $j$ of the chiral and antichiral symmetries, respectively, by $Z_{ij}$. The matrix $Z$ thus describes the modular invariant torus partition function of the CFT.

We restrict our attention to conformal defects that preserve enough additional symmetry to keep the model rational. We call a defect ‘simple’, iff it cannot be written as a sum of other defects. The number of simple defects is given by $\text{tr}(Z^2)$ [5, 3]. Let us denote the set of simple defects by $\{D_\alpha | \alpha \in \mathcal{K}\}$ for some label set $\mathcal{K}$, with the label for the trivial defect denoted by ‘$e$’. In general, one must assign an orientation to a defect line.

Consider two simple defects running parallel to each other and with the same orientation. In the limit of vanishing distance they fuse to a single defect which is, in general, a superposition of simple defects. This gives rise to a (not necessarily commutative) fusion algebra of defects [5, 11], written schematically as

$$D_\alpha \otimes D_\beta = \sum_{\gamma \in \mathcal{K}} \tilde{N}_{\alpha \beta}^\gamma D_\gamma .$$

In the TFT formalism, the general class of models we are studying now is described by an algebra $A$ in the category of representations of the chiral algebra of the CFT. Defects are then described as bimodules of $A$, and the defect fusion rules above amount to decomposing the tensor product over $A$ of two bimodules into a direct sum of simple bimodules, which can be performed explicitly. The bimodule describing the trivial defect $D_e$ turns out to be $A$ itself. If the two parallel defects have opposite direction we write $D_\alpha \otimes D_\beta'$. Two subsets of defects turn out to be of particular interest. The first one is the set $\mathcal{G}$ of group-like defects. A defect $X$ is called group-like, iff $X \otimes X' = D_e$. One can show that group-like defects are simple, so that $\mathcal{G} \subseteq \mathcal{K}$. Further, for two group-like defects $D$ and $D'$, their fusion $D \otimes D'$ is again group-like. This turns $\mathcal{G}$ into a (in general nonabelian) group with unit $D_e$, via $D_\alpha \otimes D_\beta = D_\gamma$ and $D_\gamma^{-1} = D_{\gamma'}$. From figure 3 we see that taking any group-like defect past a bulk field results in a sum of bulk fields, since the only intermediate defect that does occur is the trivial one, $D_\gamma \otimes D_{\gamma'} = D_\varepsilon$. Commuting a group-like defect past all bulk fields in a correlator results in a correlator of different bulk fields, but having the same value. Thus, group-like defects produce an internal symmetry of the CFT. For the Ising model one has $\mathcal{G} = \{1, \varepsilon\}$, a $\mathbb{Z}_2$ group, and from figure 3 we see that the defect $\varepsilon$ indeed acts by reversing the sign of the spin field.

The second and larger subset is formed by the duality defects. A defect $X$ is a duality defect, iff there exists another defect $Y$ such that taking first $X$ and then $Y$ past a bulk field results only in a sum of bulk fields, with no disorder fields present. In other words, commuting $X$ past all fields in an order correlator in general gives
The critical three-states Potts model

The critical three-states Potts model has central charge $c = 4/5$ and corresponds to a $D$-type model in the classification of Virasoro-minimal models. It has first been considered in [12]. The number of simple conformal defects in this model is $tr(ZZ') = 16$ (and there are 8 conformal boundary conditions). The defect fusion rules can be computed using Ocneanu quantum algebras [5, 11], or weak Hopf algebras, or by TFT methods. The result can be summarised as follows. The set of defect labels can be written as $\mathcal{G} = \mathcal{K} \times \mathcal{K}$, with $\mathcal{K}_x = S_3 \cup \{u_+, u_-=1\}$ and $\mathcal{K}_y = \{1, \varphi\}$, where $S_3$ denotes the permutation group of three symbols. The fusion product $D_{x,y} \otimes D_{x',y'} = \sum_{x''} x'' | x'' \in \mathcal{K}_x \times \mathcal{K}_y$ is given by the following rules. The product in $\mathcal{K}_y$ is given by Lee-Yang fusion rules $\varphi \cdot \varphi = 1 + \varphi$, while the product in $\mathcal{K}_x$ is described as follows. For $p, p' \in S_3$, $p \cdot p'$ is given by the product in $S_3$, and $p \cdot u_+ = u_+$, where $\varepsilon \in \{\pm 1\}$ and $\varepsilon' = \varepsilon \operatorname{sgn}(p)$; finally, denoting the elements of $S_3$ by $e$ (identity), $p_{12}$, $p_{13}$, $p_{23}$ (transpositions), and $p_{123}$, $p_{132}$ (cyclic permutations), we have $u_+ \cdot u_+ = u_-$, $u_+ \cdot u_- = e + p_{123} + p_{132}$ and $u_- \cdot u_+ = u_- \cdot u_+ = p_{12} + p_{132} + p_{23}$. Owing to the presence of $S_3$, the fusion algebra of defects is non-commutative in this model. One can convince oneself that the group-like defects are $\mathcal{G} = \{(p, 1) \mid p \in S_3\}$ and the duality-defects are $\mathcal{D} = \{(x, 1) \mid x \in \mathcal{K}_x\}$.

The $S_3$-structure of the group-like defects could again have been expected from the lattice model realisation of the three-states Potts model; it amounts to a permutation of the three possible values of the spin.

The critical three-states Potts model contains 12 primary bulk fields and 208 primary disorder fields. Of these, we consider the energy operator $E(z)$ of left/right conformal weight $\left(\frac{c}{2}, \frac{c}{2}\right)$, the two spin fields $S_{\pm}(z)$ of weight $\left(\frac{c}{12}, 0\right)$ and the two disorder fields $Z_{\pm}(z)$ of the same weight, where $Z_+$ generates a defect of type $(p_{123}, 1)$ and $Z_-$ one of type $(p_{132}, 1)$. We find that taking a duality defect of type $(n_z, 1)$ through a spin field $S_3(n_z)$, for $\varepsilon, \nu \in \{\pm 1\}$, generates a disorder field $Z_{\varepsilon\nu}(z)$, and vice versa. Furthermore, taking $D_{u_+1}$ past the energy field $E(z)$ gives $-E(z)$, so that the order/disorder duality at the critical point extends to a high/low temperature duality off the critical point.

Conclusions

We have demonstrated that the fusion algebra of defects in a CFT contains a lot of physical information: Internal symmetries correspond to group-like defects, and the order/disorder dualities to duality defects. The analysis is carried out within CFT, it allows one to study symmetry properties of universality classes of critical behaviour without reference to a particular lattice realisation. To compute the dual correlator one must simply commute a given duality defect past any field insertions. This procedure can be applied to correlators on surfaces of arbitrary genus and even with boundary. Via conformal perturbation theory, one can also identify high/low temperature dualities in the vicinity of the critical point.

To conclude, we mention that these considerations can also be applied to the free boson. One then finds that $T$-duality is induced by duality defects, too. The defect line in this example is labelled by the $\mathbb{Z}_2$-twisted representation of the $U(1)$-current algebra.