Algebra in Braided Tensor Categories and Conformal Field Theory

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An introduction and overview for the works

[I] J. Fuchs, I. Runkel, C. Schweigert, *TFT construction of RCFT correlators. I: Partition functions*, Nucl. Phys. B646 (2002) 353–497, `hep-th/0204148`.

[II] J. Fuchs, I. Runkel, C. Schweigert, *TFT construction of RCFT correlators. II: Unoriented world sheets*, Nucl. Phys. B678 (2004) 511–637, `hep-th/0306164`.

[III] J. Fuchs, I. Runkel, C. Schweigert, *TFT construction of RCFT correlators. III: Simple currents*, Nucl. Phys. B694 (2004) 277–353, `hep-th/0403157`.

[IV] J. Fuchs, I. Runkel, C. Schweigert, *TFT construction of RCFT correlators. IV: Structure constants and correlation functions*, Nucl. Phys. B715 (2005) 539–638, `hep-th/0412290`.

[C] J. Fröhlich, J. Fuchs, I. Runkel, C. Schweigert, *Correspondences of ribbon categories*, `math.CT/0309465`, Adv. Math. 199 (2006) 192–329.

Contents

1 Introduction

The works [I]–[IV] combine algebra in braided tensor categories and topological field theory in three dimensions to construct correlation functions of two-dimensional euclidean conformal quantum field theories. In [C] some relevant aspects of the representation theory of algebras in braided tensor categories are investigated in depths. The present text provides an introduction and overview of [I]–[IV] and [C] and places these works into context.

Classical conformal invariance

Recall that two $\mathcal{C}^\infty$-manifolds M, M' with metrics g, g' (either of euclidean or Minkowski signature) are *conformally equivalent* if there is a diffeomorphism $f : M \to M'$, called *conformal transformation*, such that $(f^*g')(p) = \Omega(p)g(p)$ for some smooth function $\Omega : M \to \mathbb{R}_{>0}$. In words, conformal transformations preserve angles, but not necessarily lengths.

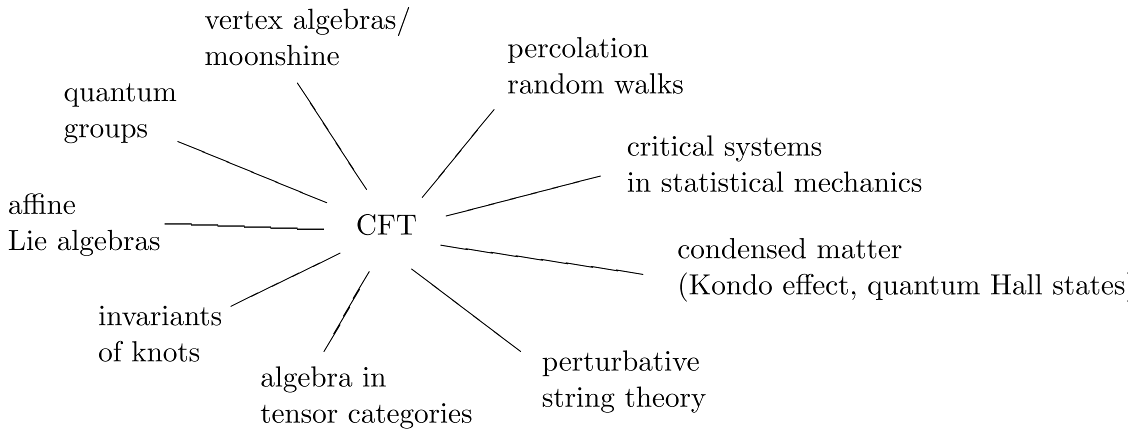
An example of a classical field theory with conformal invariance is a free scalar field in two dimensions. It can be formulated in terms of an action principle for smooth functions ϕ from a two-manifold M with metric g to $\mathbb{R}$,

$$S_g[\phi] = \int_M \Big(g^{ij} \tfrac{\partial}{\partial x^i}\phi \, \tfrac{\partial}{\partial x^j}\phi \Big) d\mathrm{vol} \quad , \tag{1.1}$$

This action is invariant under Weyl-transformations of the metric, i.e. $S_g[\phi] = S_{g'}[\phi]$ if the metrics g and g' on M are related by $g(p) = \Omega(p)g'(p)$ for some $\Omega : M \to \mathbb{R}_{>0}$. In particular, the field theory (1.1) has conformal symmetry. The most famous example of a classical field theory with conformal invariance is Maxwell's theory of electrodynamics.

Conformal quantum field theories

The study of quantum field theories with conformal symmetry emerged in the late 1960s on the one side from the study of critical behaviour in statistical mechanics [Py], and on the other side from investigations of the high energy behaviour of quantum field theories and the renormalisation group [Wl]. In two dimensions, an interacting quantum field theory which exhibited conformal symmetry was presented by Thirring already in 1958 [Th]. The major breakthrough came with the realisation by Belavin, Polyakov and Zamolodchikov in 1984 that in a certain class of 2dCFTs, which is now called the Virasoro minimal models, the correlators can be found by solving linear differential equations [BPZ]. Since then conformal field theory has developed many more connections to various areas in mathematics and physics,



The present text makes use of the connection to the invariants of knots and three-manifolds (via three-dimensional topological field theory, see chapter 3), to vertex algebras (chapter 4) and to algebra in tensor categories (chapter 5). The relevance of the latter to euclidean CFT was discovered and announced in [FuRS] and elaborated in the works [I]–[IV] and [C].

In view of the above diagram, it would be important to formulate an axiomatic framework for 2dCFT in order to have a well-defined setting in which to study its properties, as well as to develop methods which allow to construct examples. There are at present two rather different axiomatic approaches to 2dCFT, depending on whether one considers Minkowskian or euclidean theories. While in the latter case an all-encompassing axiomatic framework is not yet available, for theories in Minkowski space one can apply the formulation of algebraic quantum field theory.

1.1 Conformal field theory in Minkowski space

The approaches to axiomatic QFT in d-dimensional Minkowski space M are first, the formulation via fields inserted at points in terms of the Wightman axioms [SW] and second, the formulation via algebras of observables related to regions of M in terms of algebraic QFT (also called Local Quantum Physics) by Araki, Haag and Kastler [Ha]. We will briefly introduce some concepts relevant for the latter.

Denote by $\eta(x,y) = x_0y_0 - \sum_{i=1}^{d-1} x_iy_i$ the metric on M. A double cone O is the intersection of a forward and a backward light cone $V_\pm = \{x \in M \,|\, \eta(x,x)>0\,,\ \pm x_0>0\,\}$ in M, i.e. $O = (V_+{+}x) \cap (V_-{+}y)$ for some $x,y \in M$. Let K be the set of double cones in M. A *net of von Neumann algebras* is an inclusion preserving assignment $O \mapsto \mathcal{A}(O)$ where $O \in K$ and the $\mathcal{A}(O)$ are von Neumann algebras on a common Hilbert space $\mathcal{H}$. In QFT, these are the 'algebras of observables on the space-time region O'. In the application to QFT, a net $\mathcal{A}$ of von Neumann algebras has to be covariant and local. The net $\mathcal{A}$ is called *covariant*, iff each element g of the Poincaré group gives rise to a family $\alpha_{g,O} : \mathcal{A}(O) \to \mathcal{A}(gO)$ of automorphisms of C^*-algebras (note that if $O \in K$ then so is gO). Further, $\mathcal{A}$ is called *local* iff $\big[\mathcal{A}(O_1), \mathcal{A}(O_2)\big] = \{0\}$ whenever O_1 and O_2 are spacelike separated (i.e. $g(x_1{-}x_2, x_1{-}x_2) < 0$ for all $x_1 \in O_1$ and $x_2 \in O_2$).

The relation to tensor categories appears in the study of 'superselection sectors' of a local, covariant net $\mathcal{A}$ [DHR, DR1], that is, of appropriately defined representations (positive energy representations satisfying the DHR-criterion) of $\mathcal{A}$. For $d \geq 4$ space-time dimensions, the category of such representations is a symmetric tensor category, which contains a subcategory equivalent to the category G-mod of finite dimensional continuous unitary representations of a compact group G. The group G gives the global symmetries of the QFT. In fact, from the knowledge of the superselection sectors one can recover the symmetry group [DR2].

In two [FhRS] and three [FG1] dimensions one finds braid statistics, i.e. the category formed by the superselection sectors is still braided, but in general no longer symmetric. We will concentrate on the application of algebraic QFT to chiral CFT in two-dimensional Minkowski space [Ma, FG2], following the review in [Mu1].

The restriction to chiral CFT means that one considers only 'left-moving degrees of freedom' (say), so that the net $\mathcal{A}$ on two-dimensional Minkowski space can be recovered from its restriction to the line R given by $x_0{=}0$, in the following sense. Each double cone $O \in K$ can be projected to R along the light ray $x_0{=}x_1{+}(\text{const})$, giving an open interval I on R. If two double-cones O_1 and O_2 result in the same interval I, then the associated von Neumann algebras coincide, $\mathcal{A}(O_1) = \mathcal{A}(O_2)$. In more detail, a chiral CFT is defined as follows.

Let L be the set of intervals on S^1 (the compactification of the line R above), that is, the set of connected open non-dense subsets of S^1. A chiral conformal field theory in Minkowski space consists of a Hilbert space $\mathcal{H}_0$ with a distinguished vector Ω (the vacuum), an assignment $I \mapsto \mathcal{A}(I)$ of von Neumann algebras to intervals forming a net, and a strongly continuous unitary representation U of the Möbius group $\mathrm{PSU}(1,1)$ on $\mathcal{H}_0$. The net $\mathcal{A}$ has to be local ($\mathcal{A}(I) \subset \mathcal{A}(J)'$ if $I \cap J = \emptyset$) and covariant ($U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$), as well as irreducible, with a unique vacuum and the representation U has to be of positive energy (see [FG2, Mu1] for details). A large class of examples of chiral CFTs can be constructed in terms of representations of loop groups [BMT, FG2, Wa].

A representation π of the net $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\pi$ is a family $\{\, \pi_I \,|\, I \in L \,\}$, where each π_I is a representation of $\mathcal{A}(I)$ on $\mathcal{H}_\pi$, and for $I \subset J$ we have $\pi_J\big|_{\mathcal{A}(I)} = \pi_I$. Denote by $\mathrm{Rep}(\mathcal{A})$ the category of separable, irreducible representations of $\mathcal{A}$, completed w.r.t. direct sums (see [Mu1] for details). If a chiral CFT in Minkowski space $\mathcal{A}$ satisfies three more properties, namely it has to be strongly additive, split and of finite index (for details refer again to [Mu1]), it is called *completely rational*. For a completely rational CFT in Minkowski space one can prove [KLM] that $\mathrm{Rep}(\mathcal{A})$ is a modular tensor category (the definition is reviewed in section 3.2) which is in addition unitary.

In the study of extensions of local nets of both, chiral theories (where the net is over the line R) and full theories (where the net is over the Minkowski space M) one needs the notion of 'nets of subfactors' [LRe1]. A *factor* is a von Neumann algebra B with trivial centre, and a *subfactor* is a von Neumann algebra A which is a factor, as well as a subalgebra of B which has the same unit as B. For a net of subfactors, one has a subfactor $\mathcal{A}(O) \subset \mathcal{B}(O)$ for every interval (in the chiral case), respectively every double cone (in the full theory), O, see [LRe1] for more details. A subfactor $A \subset B$ can alternatively be characterised by a so-called 'Q-system' in B [Lo, LRo]. Similarly, a Q-system can characterise the extension $\mathcal{A} \subset \mathcal{B}$ of a local net [LRe1, Re].

The approach of algebraic QFT has also been applied to boundary conformal field theory on a two-dimensional Minkowski half-space $\{\, (x_0, x_1) \,|\, x_1{\geq}0 \,\}$ [LRe2]. Further, in [BFV] the formalism of algebraic QFT is extended to space-times with Lorentzian metric.

1.2 Conformal field theory in euclidean space

In this text we will be concerned with euclidean CFTs (from hereon also referred to only as CFTs) that can be defined on two-dimensional surfaces of arbitrary genus. From the point of view of applications to statistical mechanics or condensed matter systems, this may seem a somewhat unnatural restriction. On the other hand, in the application to string theory, it is necessary to control the CFT also on surfaces of higher genus.

As already mentioned, up to now there is no universally agreed list of axioms which can be taken as "the" axioms defining a CFT, and which covers all known models one might want to call a CFT. There exist, however, precise mathematical frameworks for certain aspects of CFT.

There are, broadly speaking, two related languages in which one can formulate the properties a CFT should fulfil. In the field theoretically motivated language one uses correlation functions assigned to surfaces with point like field insertions and poses conditions on their behaviour when two insertion points are taken close to each other. This is the point of view used in the seminal work [BPZ]. If one restricts oneself to so-called *holomorphic fields* (see section 2.4) one obtains the mathematical notion of a meromorphic CFT [Go, GG] and the notion of a conformal vertex algebra, originally due to Borcherds [B] and by now subject of several books [FLM, Kc, Hu, LL, FB].

In the formulations motivated by string theory (see e.g. [FrS, Va] or [Pl, section 9.4]) one would instead assign maps to surfaces with holes and require properties for the behaviour of these maps under cutting and gluing, an idea which has been cast into the language of functors by Segal [Se1, Se2]. This approach to CFT has been reviewed and developed further e.g. in [Ga1, Ga2, HK1, HK2], and it is also the formulation we will use in most of chapter 2.

Comparing to the formulation of CFT in Minkowski space in terms of algebraic QFT, a conformal vertex algebra is the analogue of a chiral CFT in Minkowski space, while a euclidean CFT on a surface of genus zero corresponds to a full CFT in Minkowski space. The difficulty in finding a good set of axioms resides in the need to formulate the euclidean CFT also on surfaces of higher genus.

It is not the aim of this text or of the works [I]–[IV],[C] to provide an axiomatic definition of a CFT. The description in section 2.4 is intended to show what one aims for, rather than to be the final answer. Instead, these works are part of a larger research effort to develop the methods necessary to gain complete control over a large class of examples for CFTs, the so-called *rational conformal field theories*. This research effort can be broadly divided in two parts, a "bottom up", or *complex-analytic* part, and a "top down", or *algebraic* part.

In the complex-analytic part one treats the *chiral conformal field theory*, that is, one formalises the properties of holomorphic fields in the notion of a conformal vertex algebra. Chiral conformal field theory should be thought of as encoding the symmetries of a CFT. The study of representations of a vertex algebra then gives two pieces of information. First, the space of all fields of the CFT has to be a representation of the vertex algebra, so that one obtains constraints on the field content of the CFT. Second, it provides the so-called *conformal blocks*, multi-valued analytic functions which serve as the basic building blocks of the correlation functions of the CFT. We will return to this point in chapter 4.

In the algebraic part one is concerned with the *full conformal field theory*. Here one takes the analysis of the chiral conformal field theory as an input and tries to assemble the conformal blocks into a system of correlators that fulfils the consistency conditions required for a CFT. For a general vertex algebra, this problem is still too hard to solve. We will restrict our attention to a class of vertex algebras which are more manageable, and refer to those as *rational chiral conformal field theories*. In short, we demand that the representation category of the vertex algebra is a modular tensor category (section 3.2) and that the 3dTFT derived from it (section 3.4) correctly encodes the factorisation and monodromy of the conformal blocks (section 4.3). Given such a rational chiral CFT and the associated modular tensor category, one can answer the question "What is a consistent system of correlation functions?" (Problems 6.4 and 6.6) purely on the level of this tensor category, without further reference to the often rather complicated representation theory and spaces of conformal blocks of the associated vertex algebra.

This is the starting point of the treatment in [I]–[IV]. It is shown that a symmetric special Frobenius algebra leads to a solution of the consistency conditions for CFT correlators (Theorem 6.11). Properties of these algebras are described in chapter 5. The basic tool used in the construction of CFT correlators, three-dimensional topological field theory, is reviewed in chapter 3. Finally, the construction of the correlators is described in chapter 6.

The works [I]–[IV] are thus an important step in the construction of CFTs since they solve the second, algebraic, part in the program outlined above for the class of rational CFTs.

The investigation of the chiral CFT was termed "bottom up" because it starts from a subset of the correlators of the CFT, which is then used to constrain which form the full set of correlators can take. The algebraic part was called "top down" because it takes a rather sophisticated piece of information as an input, the monodromy and factorisation properties of conformal blocks as encoded in a modular tensor category, and uses this to determine which combinations of conformal blocks describe correlation functions of a CFT. What is still missing is the final link, i.e. a precise list of properties for a vertex algebra to be a rational chiral CFT in the above sense, so that it can serve as an input for the algebraic construction. This is an important goal for future investigations.

1.3 Frobenius algebras and tensor categories

It should be appreciated that the same structure, a modular tensor category, appears in the study of the chiral theories in both, Minkowskian and euclidean conformal field theories. In fact, the structural similarity between the two approaches extends even further, because also in the study of subfactors, Frobenius algebras arise naturally, in the guise of 'Q-systems' [Lo, LRo]. Indeed, every Q-system is a symmetric special *-Frobenius algebra [EP]. As mentioned in section 1.1, Q-systems characterise extensions of nets of von Neumann algebras. The relevance of symmetric special Frobenius algebras to the computation of correlators in boundary CFT was first pointed in [FuRS]. With these considerations in mind, it is a natural aim to investigate the properties of such algebras in tensor categories.

Algebras in symmetric tensor categories already played an important role in Deligne's characterisation of Tannakian categories (see e.g. [Sa, DM]). They were studied in much detail by Pareigis (see e.g. [Pa1, Pa3]). More recently, commutative algebras were e.g. studied in the context of conformal field theory and quantum subgroups in [KO], in relation to weak Hopf algebras in [Os], and in connection with Morita equivalence for tensor categories in [Mu2]. The algebras relevant in the conformal field theory context are symmetric special Frobenius algebras [FuS, FuRS, I]; those encoding properties of conformal field theory on surfaces with boundary are, generically, non-commutative.

In [C], such aspects of the representation theory of algebras in braided tensor categories are investigated, which have no nontrivial classical analogue, i.e., when applied to the category of vector spaces (or any symmetric tensor category), these results become tautologies.

A commutative algebra A in a braided tensor category has an interesting subclass of A-modules, the so-called dyslectic [Pa6], or local, modules (section 5.3); when specialising to symmetric tensor category, every A-module becomes local. Let now A be not necessarily commutative. One can then distinguish two different centres of A [VZ, Os], the left centre C_l and right centre C_r (section 5.2). If the braiding is symmetric, the left and right centre coincide. However, in the genuinely braided case, they can be non-isomorphic (as illustrated in Example 5.13 below). Nonetheless, as the first main result in [C], the category of local C_l-modules is equivalent to the category of local C_r-modules (Theorem 5.20).

As another example, consider correspondences of finite groups. A correspondence of two groups G_1 and G_2 is a subgroup R of $G_1 \times G_2$. One can now wonder if, given the categories $\mathcal{R}ep(G_1)$ and $\mathcal{R}ep(R)$ of finite-dimensional complex representations of G_1 and R, one can recover $\mathcal{R}ep(G_2)$. It is possible to find a commutative algebra A in $\mathcal{R}ep(G_1) \boxtimes \mathcal{R}ep(G_2)$ (the product $\boxtimes$ is defined in section 6.1 of [C]) such that the category of A-modules is equivalent to $\mathcal{R}ep(R)$. The original question can then rephrased as, given $\mathcal{R}ep(G_1)$ and the category of modules of a commutative algebra in $\mathcal{R}ep(G_1) \boxtimes \mathcal{R}ep(G_2)$, can one recover $\mathcal{R}ep(G_2)$? Clearly, the answer is "no", as can be seen by taking G_1 and R to be trivial. Surprisingly, in a truly braided setting, an analogous problem can be solved (Theorem 5.23). This constitutes the second main result of [C].

This text is organised as follows. Chapters 2–4 provide an introduction and background to the problem we ultimately want to treat, namely the solution of the algebraic part in the two-step construction of a CFT. In these chapters, emphasis has been laid on conveying the general ideas, rather than on a detailed derivation (which is also not always available). The purpose of chapters 2–4 is to motivate the questions addressed in chapters 5 and 6, which then give an overview of the main results in [I]–[IV] and [C]. There, care has been taken to properly define all the notions needed in the statements of the main theorems.

Sections, definitions, equations etc., of [I]–[IV] and [C] will be referred to as section II:2.3, Definition C:3.20, equation (IV:5.47) and so forth.

2 Two-dimensional conformal field theory

It is a recurring theme in this text that certain quantum field theories are expressed as functors. This is physically motivated by the euclidean path integral, and by its discrete version, a statistical lattice model.

In this chapter we will treat two topological QFTs (sections 2.1 and 2.3) as well as one lattice model (section 2.2). These should motivate the functorial formulation of euclidean 2dCFT in section 2.4. In chapters 4 and 6, the formulation of CFT in terms of correlators is used; this is reviewed in sections 2.5 and 2.6.

2.1 Two-dimensional topological field theory

A simple but instructive example of a quantum field theory that is expressed as a functor is that of a two dimensional topological quantum field theory (2dTFT). This axiomatic framework was first discussed in [At1]; detailed expositions can be found e.g. in [Q, Ko] or [BK, section 4.3].

A 2dTFT is a tensor functor from the cobordism category $\mathsf{2Cob}$ to the finite dimensional $\Bbbk$-vector spaces $\mathcal{V}ect_f(\Bbbk)$, for some field $\Bbbk$. An object U in $\mathsf{2Cob}$ is either an ordered disjoint union of oriented circles S^1, or the empty set. A morphism $m : U \to V$ is an equivalence class of oriented, compact two-manifolds with parametrised boundary (M, ι, o). Here $\iota : U \to \partial M$ (standing for "in") and $o : V \to \partial M$ (standing for "out") are injective, continuous, have non-intersecting images and cover the boundary of M, $\partial M = \mathrm{Im}(\iota) \cup \mathrm{Im}(o)$; the map ι is orientation preserving while o reverses the orientation (∂X is oriented via the inward pointing normal). The equivalence relation $(M, \iota, o) \cong (M', \iota', o')$ on cobordisms is given by orientation preserving homeomorphisms that respect the boundary parametrisation. The composition $U \xrightarrow{m} V \xrightarrow{m'} W$ is given by gluing the two-manifolds using the parametrisation of their boundaries. The unit morphism $id_U : U \to U$ is provided by (the equivalence class of) the unit cylinder $U \times [0, 1]$ over U. $\mathsf{2Cob}$ is a strict tensor category,[1] with the tensor product given by disjoint union of objects and morphisms, and unit object $\mathbf{1}$ being the empty set. Furthermore, $\mathsf{2Cob}$ is equipped with a *partial trace* (or cancellation, cf. [HK1]), i.e. for any objects U, V, W we have a map $\mathrm{tr}^{(W)} : \mathrm{Hom}(U \otimes W, V \otimes W) \to \mathrm{Hom}(U, V)$, which acts as follows on $m \in \mathrm{Hom}(U \otimes W, V \otimes W)$. Choose a representative (M, ι, o) of m and construct a new cobordism M' by identifying $\iota(W) \cong o(W)$ (the in- and out-going boundary components labelled by W are glued together). Then $\mathrm{tr}^{(W)}(m) : U \to V$ is the equivalence class of M'.

We will use the notation $(Z, \mathcal{H})$ for the functor $\mathsf{2Cob} \to \mathcal{V}ect_f(\Bbbk)$. Here $\mathcal{H}$ denotes the action of the functor on objects and Z the action on morphisms. $(Z, \mathcal{H})$ is required to preserve the partial trace in the sense that $Z\big(\mathrm{tr}^{(W)}(m)\big) = \mathrm{tr}^{(\mathcal{H}(W))} Z(m)$.

Recall that a Frobenius algebra over a field $\Bbbk$ is a pair (A, ε) where A is an algebra over $\Bbbk$ and ε is a linear map $A \to \Bbbk$, called *trace*, with the property that the bilinear, invariant form $b(a, b) = \varepsilon(a \cdot b)$ on $A \times A$ is non-degenerate. In particular, being Frobenius is an additional structure, not a property of an algebra. An equivalent characterisation of Frobenius algebras will be given in Theorem 2.3 below.

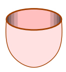
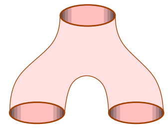
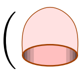
It turns out that 2dTFTs are in fact the same as finite dimensional, commutative Frobenius algebras.

Theorem 2.1:
Let $\Bbbk$ be a field. The 2dTFTs $(Z, \mathcal{H}) : \mathsf{2Cob} \to \mathcal{V}ect_f(\Bbbk)$ are in one-to-one correspondence with finite dimensional commutative Frobenius algebras A over $\Bbbk$.

This theorem is taken from [BK, section 4.3], it is originally due to [Ab1] (where it is formulated as an equivalence of the category of 2dTFTs with the category of commutative Frobenius algebras). Related earlier results can be found in [D, Vo].

[1] Often, the existence of a duality is included in the definition of a tensor category. What we refer to as a tensor category is then called a *monoidal* category.

To see the idea of the proof, suppose we are given a 2dTFT $(Z, \mathcal{H})$. Then we set $A = Z(S^1)$. The unit, multiplication and trace are given by applying Z to the following cobordisms

$\eta = Z\Big($  $\Big) : \Bbbk \to A$, $\quad m = Z\Big($  $\Big) : A \otimes A \to A$, $\quad \varepsilon = Z\Big($  $\Big) : A \to \Bbbk$. (2.1)

Here the unit $e \in A$ is encoded in the linear map $\eta : \Bbbk \to A$ s.t. $\eta(1) = e$. Associativity, unit-property and non-degeneracy of ε then follow from functoriality of Z and comparing the glued cobordisms. Conversely, one can construct a functor $(Z, \mathcal{H})$ given a commutative Frobenius algebra.

Remark 2.2:
The 2dTFT described above is a *closed* TFT because the only boundaries a cobordism is allowed to have are parametrised boundaries linked to an object of $\mathsf{2Cob}$. There is also an open/closed version of 2dTFT [La, Mo]. In this case one considers a cobordism category whose objects are disjoint unions of intervals and circles, and whose corbordisms are manifolds with boundaries and corners. An open/closed TFT then corresponds to a not necessarily commutative Frobenius algebra.

2.2 Lattice models as a functor

A lattice model can be thought of as a discrete version of a euclidean quantum field theory. It will also be described by a functor, with cobordism now given by cell-complexes.

The category $\mathsf{2Cpx}$ is defined as follows. Denote by D_n an oriented polygon with $n>0$ vertices and edges, as well as a preferred vertex labelled 1. An object U of $\mathsf{2Cpx}$ is an ordered disjoint union $D_{n_1} \sqcup \cdots \sqcup D_{n_k}$, or the empty set. A morphism $L : U \to V$ is a triple $L = (\Gamma, \iota, o)$. Here Γ is a two dimensional (abstract) oriented cell complex, i.e. we have sets of vertices $v(\Gamma)$, edges $e(\Gamma)$ and faces $f(\Gamma)$. Further, ι is an orientation preserving injection $\iota : U \to \partial\Gamma$ and o an orientation reversing injection $o : U \to \partial\Gamma$ such that $\mathrm{Im}(\iota) \cap \mathrm{Im}(o) = \emptyset$ and $\partial\Gamma = \mathrm{Im}(\iota) \cup \mathrm{Im}(o)$. Composition of $U \xrightarrow{L} V \xrightarrow{L'} W$ is given by identifying the edges and vertices via the maps o of L and ι of L'.

Since a cell complex Γ with a non-empty boundary has at least one face, composing morphisms always increases the number of faces. Thus $\mathsf{2Cpx}$ is actually a *non-unital* category (a notion taken from [Mi]), i.e. a category with associative composition, but without unit morphisms.

Similar to the previous example, $\mathsf{2Cpx}$ becomes a strict tensor category by taking the tensor product to be given by disjoint union of objects and morphisms. There is also a trace $\mathrm{tr}^{(W)}$: $\mathrm{Hom}(U \otimes W, V \otimes W) \to \mathrm{Hom}(U, V)$, which acts on a morphism $L = (\Gamma, \iota, o)$ by replacing Γ with a new cell-complex Γ' obtained by identifying $\iota(W)$ with $o(W)$, s.t. $\mathrm{tr}^{(W)}(L) = (\Gamma', \iota|_U, o|_V)$.

Lattice models from statistical mechanics provide examples of tensor functors $(Z, \mathcal{H})$: $\mathsf{2Cpx} \to \mathcal{V}ect_f(\Bbbk)$. Let us illustrate this in the case of the Ising model. On the objects D_n we set

$$\mathcal{H}(D_n) = \mathrm{span}_{\Bbbk}\big\{\sigma : v(D_n) \to \{-1, +1\}\big\} \quad , \tag{2.2}$$

that is, the 2^n-dimensional vector space spanned by maps from the vertices of the polygon D_n to the two element set $\{-1,+1\} \subset \mathbb{Z}$. For a general object in $\mathsf{2Cpx}$ we take appropriate tensor products of the spaces (2.2). In physical terms, $\sigma(i)$, $i \in v(D_n)$ is the *spin* at the lattice site i. The vector space $\mathcal{H}(D_n)$ is called the *space of states* on D_n.

Choose a constant $q \in \Bbbk^\times$. This will be a parameter entering the definition of the lattice model. In statistical mechanics one takes $\Bbbk = \mathbb{C}$ and $q = e^{-\beta}$ where β is the inverse temperature. On $\mathcal{H}(D_n)$ we define a non-degenerate pairing $(\,\cdot\,,\,\cdot\,)_n$ in terms of the values it takes on basis vectors $\sigma, \tau : v(D_n) \to \{-1,1\}$ as

$$(\sigma,\tau)_n \;=\; \delta_{\sigma,\tau} \prod_{\langle i,j\rangle \in e(D_n)} q^{\sigma(i)\sigma(j)} \;=\; \delta_{\sigma,\tau} \prod_{i=1}^{n} q^{\sigma(i)\sigma(i+1)} \quad . \tag{2.3}$$

Here the product $\langle i,j\rangle \in e(D_n)$ is over all edges in D_n; i and j denote the vertices at the ends of the edge. In particular we can write

$$id_{\mathcal{H}(D_n)} = \sum_\tau \tau \,\frac{(\tau,\cdot)_n}{(\tau,\tau)_n} \quad . \tag{2.4}$$

On the spaces $\mathcal{H}(U)$ for a general object U of $\mathsf{2Cpx}$ the non-degenerate pairing $(\,\cdot\,,\,\cdot\,)_U$ is defined analogously.

Given a morphism $L : U \to V$, to fix $Z(L)$ it is enough to define $(\tau, Z(L)\sigma)_V$ for all basis elements σ, τ. Suppose $L = (\Gamma, \iota, o)$. We set

$$(\tau, Z(L)\sigma)_V = \sum_s \prod_{\langle i,j\rangle \in e(\Gamma)} q^{s(i)s(j)} \quad . \tag{2.5}$$

The sum over s is over all maps $s : v(\Gamma) \to \{-1,1\}$ with values on the boundary $\partial\Gamma$ fixed by the conditions $s(\iota(i)) = \sigma(i)$ for all vertices $i \in v(U)$ and $s(o(j)) = \tau(j)$ for all vertices $j \in v(V)$. The quantity (2.5) is called *partition function* or *state sum* in statistical mechanics.

As an illustration, let us verify that the so defined $Z(L)$ is consistent with composition. Consider morphisms $U \xrightarrow{L} V \xrightarrow{L'} W$ with $L = (\Gamma, \iota, o)$ and $L' = (\Gamma', \iota', o')$. Using (2.4), on the one hand we have

$$\begin{aligned}(\sigma'\,,\, Z(L')Z(L)\,\sigma)_W &= \sum_\tau \frac{(\sigma', Z(L')\tau)_W\;(\tau, Z(L)\sigma)_V}{(\tau,\tau)_V} \\ &= \sum_\tau \frac{1}{(\tau,\tau)_V} \sum_{s,s'} \prod_{\langle i,j\rangle \in e(\Gamma)} q^{s(i)s(j)} \prod_{\langle k,l\rangle \in e(\Gamma')} q^{s'(k)s'(k)} \quad .\end{aligned} \tag{2.6}$$

The τ-sum is over all maps $\tau : v(V) \to \{-1,1\}$, the s-sum over all maps $s : v(\Gamma) \to \{-1,1\}$ with boundary values given by τ and σ, and finally the s'-sum is over all maps $s' : v(\Gamma') \to \{-1,1\}$ with boundary values fixed by σ' and τ. On the other hand, for $L' \circ L = (\Gamma'', \iota'', o'')$,

$$(\sigma'\,,\, Z(L' \circ L)\,\sigma) \;=\; \sum_{s''} \prod_{\langle i,j\rangle \in e(\Gamma'')} q^{s''(i)s''(j)} \quad , \tag{2.7}$$

where s'' is summed over all maps $s'' : v(\Gamma'') \to \{-1,1\}$ with boundary values given by σ' and σ. One can now convince oneself that by construction of Γ'', the sum over τ, s, s' in (2.6) amounts to the sum over s'' in (2.7). However, in the two products in (2.6), the edges of V appear twice, which is compensated by the factor $(\tau,\tau)_V^{-1} = \prod_{\langle i,j\rangle\in e(V)} q^{-\tau(i)\tau(j)}$.

For the Ising model, Z is obtained by summing over all possibilities to assign a spin to a vertex in $v(\Gamma)$. For other lattice models, values may be assigned also to edges or faces or combinations thereof. An example of this is provided in the next section.

2.3 Topological lattice models

In the Ising model, the linear map $Z(L)$ assigned to a morphism $L : U \to V$ depends explicitly on the cell complex Γ in $L = (\Gamma, \iota, o)$, and not only on its homotopy class. In this sense lattice models are in general not topological. However, for special choices of the state sum (2.5) the linear map $Z(L)$ only depends on the homotopy class of the complex Γ. Such a theory will be called a two-dimensional lattice TFT. After what has been said in section 2.1 it should not come as a surprise that the construction of a 2d lattice TFT also involves a Frobenius algebra.

Let us quickly recall some notions related to Frobenius algebras. One of the many alternative characterisations of a Frobenius algebra is the following [Ab2, Theorem 2.1].

Theorem 2.3:
A finite dimensional, unital, associative algebra A over a field $\Bbbk$ is Frobenius with trace $\varepsilon : A \to \Bbbk$ if and only if it has a coassociative, counital comultiplication $\Delta : A \to A\otimes A$ which is a map of A-bimodules, and which has counit ε; the A-bimodule structure on $A\otimes A$ is given by $a.(c\otimes d).b = (ac)\otimes(db)$.

A Frobenius algebra is called *symmetric* if $\varepsilon(a\cdot b) = \varepsilon(b\cdot a)$ (see e.g. [CR, p. 440]). An algebra A is called *separable* (see e.g. [Pi, KS]) if there is a map $D : A \to A\otimes A$ of A-bimodules s.t. $m\circ D = id_A$. Here $m : A\otimes A \to A$ denotes the multiplication on A. A Frobenius algebra A is called *special* [FuS, Definition 2.3] if $m\circ\Delta = \beta_A id_A$ and $\varepsilon(e) = \beta_{\mathbf{1}}$ for some constants $\beta_A, \beta_{\mathbf{1}} \in \Bbbk^\times$ and for Δ the comultiplication on A. By definition, a special Frobenius algebra is in particular separable. By modifying Δ and ε by a multiplicative factor, one can always achieve $\beta_A = 1$. For a symmetric special Frobenius algebra, $\beta_A = 1$ implies $\beta_{\mathbf{1}} = \dim(A)$, cf. [FuS, Remark 3.13]. We will always assume that for a symmetric special Frobenius algebra, coproduct and counit have been normalised in this way.

A 2d lattice TFT will again be a functor $(Z,\mathcal{H}) : \mathsf{2Cpx} \to \mathcal{V}ect_f(\Bbbk)$. For simplicity, we will only describe $Z(L)$ for the case where $L : \emptyset \to \emptyset$ and where the complex Γ in $L = (\Gamma, -, -)$ has only trivalent vertices.

Let A be a symmetric special Frobenius algebra over $\Bbbk$. Choose a basis $\{\, u_a \,|\, a\in I \,\}$ of A. Define

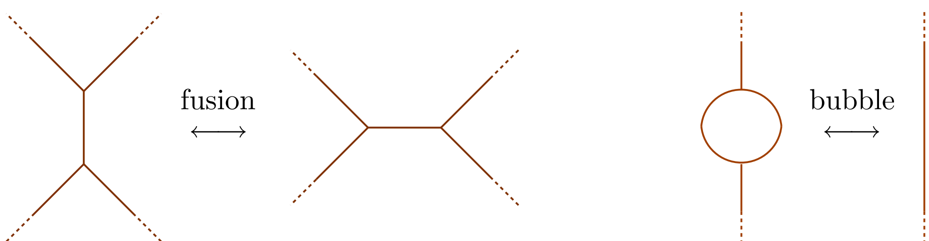
$$C_{abc} = \varepsilon(u_a u_b u_c) \qquad , \qquad g_{ab} = \varepsilon(u_a u_b) \tag{2.8}$$

and denote by g^{ab} the matrix elements of the matrix inverse to g, i.e. $\sum_b g^{ab} g_{bc} = \delta_{a,c}$. Note that since A is symmetric, C is invariant under cyclic permutation of the indices and g is symmetric. The value of $Z(L)$ will again be given as a state sum. However, rather than summing over all possibilities to assign a spin $\{-1, 1\}$ to each vertex of the 2-complex Γ as in the Ising model, we assign a value $a\in I$ to every element in the set of pairs $Q = \big\{\, (e,v)\in e(\Gamma)\times v(\Gamma) \,\big|\, v\in\partial e \big\}$.

$$Z(L) = \sum_p \prod_{v\in v(\Gamma)} C_{a(v)b(v)c(v)} \prod_{e\in e(\Gamma)} g^{a(e)b(e)} \quad , \tag{2.9}$$

where p runs over all functions $p : Q \to I$. Further, since at each vertex v three edges meet, for a given v there are three elements (e_1, v), (e_2, v), (e_3, v) in Q which share this vertex (and are ordered, up to cyclic permutations, by the 2-orientation of Γ); the values of $a(v)$, $b(v)$ and $c(v)$ in (2.9) are then defined to be $p(e_1, v)$, $p(e_2, v)$ and $p(e_3, v)$, respectively. Similarly, for a given edge e there are two pairs (e, v_1) and (e, v_2) in Q, and we set $a(e) = p(e, v_1)$ and $b(e) = p(e, v_2)$. One can verify that $Z(L)$ does not depend on the choice of basis. An explicitly basis-independent formulation is obtained by specialising the construction in chapter 6 to the category $\mathcal{C} = \mathcal{V}ect_f(\Bbbk)$.

To see that two cell complexes Γ, Γ' in the same homotopy class (or rather their associated morphisms $L, L' : \emptyset \to \emptyset$) lead to the same state sum $Z(L) = Z(L')$, it is convenient to adopt a slightly different point of view. Suppose we are given a two-dimensional compact surface Σ with $\partial\Sigma = \emptyset$. In order to assign a topological invariant to Σ, proceed in two steps. First, choose a triangulation of Σ (with three-valent vertices, and arbitrary polygons as faces). This gives a complex Γ and a morphism $L = (\Gamma, -, -)$ for which one can evaluate $Z(L)$. Second, show that this prescription is independent of the triangulation. The latter point can be established by demonstrating invariance under the 2d Matveev moves,



(2.10)

These two moves, called *fusion* and *bubble* move, allow to transform any triangulation of Σ into any other. In terms of the quantities (2.8) the moves (2.10) leave $Z(L)$ invariant if

$$\sum_{m,n} C_{ijm} g^{mn} C_{nkl} = \sum_{m,n} C_{jkm} g^{mn} C_{nli} \qquad \text{and} \qquad \sum_{m,n,p,q} C_{imn} C_{jpq} g^{mq} g^{np} = g_{ij} \tag{2.11}$$

One can verify that these identities are implied by A being symmetric special Frobenius (this is a consequence of the discussion around (I:5.11) applied to the special case $\mathcal{C} = \mathcal{V}ect_f(\Bbbk)$). Since the only topological invariant of Σ is its genus g, the state sum (2.9) should take a very simple form. Indeed, if A is in addition simple, a short calculation (e.g. by applying the general construction of CFT correlators in chapter 6 below to the category $\mathcal{V}ect_f(\Bbbk)$) shows that

$$Z(L) = \big(\dim A\big)^{1-g} \big(\dim \mathrm{Zentr}(A)\big)^{g} \quad , \tag{2.12}$$

where $\mathrm{Zentr}(A)$ is the centre of A.

2d lattice TFTs where first defined in [BP, FHK] via the state sum (2.9). Conversely, it was also shown there that for $\Bbbk = \mathbb{C}$, quantities C_{abc} and g_{ab} satisfying (2.11) as well as $C_{abc} = C_{cab}$ and $g_{ab} = g_{ba}$ define a semi-simple associative algebra over $\mathbb{C}$, i.e. a direct sum of matrix algebras. Further, for $\Bbbk = \mathbb{C}$, by Wedderburn's theorem any symmetric special Frobenius algebra is isomorphic to a direct sum of matrix algebras.

As we have just seen, a (not necessarily commutative) symmetric special Frobenius algebra A over $\Bbbk$ defines a 2d lattice TFT. On the other hand, in section 2.1 it was stated that a commutative Frobenius algebra is equivalent to a 2dTFT. The two constructions are indeed related; it can be verified that for a compact, oriented surface Σ with $\partial\Sigma = \emptyset$, we have $Z_{\text{2d lattice}}(L) = Z_{\text{2dTFT}}(\Sigma)$, where L is the morphism in $\mathsf{2Cpx}$ obtained by triangulating Σ and Z_{2dTFT} is constructed as in section 2.1 by using the commutative Frobenius algebra $\mathrm{Zentr}(A)$ (with trace given by restriction of the trace ε on A). This is easy to check for genus zero (since by convention $\varepsilon(e) = \dim A$ for the symmetric special Frobenius algebra A) and for genus one (where both sides give $\dim \mathrm{Zentr}(A)$).

2.4 2dCFT as a functor

The actual object we are interested in – a 2d CFT – can intuitively be thought of, on the one hand, as a continuum limit of a lattice model, and, on the other hand, as a generalisation of a 2dTFT. As already pointed out in the introduction, there is to date no all-encompassing axiomatic treatment of CFT, and it is also not the aim of the present text to provide such a list of axioms. The working definition described below is closely related to the approach by Segal [Se1, Se2], which has been given a precise formulation in [HK1, HK2] using "stacks of lax commutative monoids with cancellation". The two main differences are, first, that as in [Ga1, Ga2] we will consider surfaces with a metric, rather than surfaces with just a complex structure, and second, that we do not assume the state spaces to be Hilbert spaces, i.e. we will want to allow for non-unitary CFTs. The latter point is necessary, because the CFTs needed in the relation to critical percolation (recall the diagram in the introduction), or in the description of the ghost sector in string theory, are non-unitary.

A working definition

A 2dCFT $(Z, \mathcal{H})$ is a functor (with some additional properties) from the category $\mathsf{2Rie}$, where the cobordisms are two-dimensional Riemannian manifolds, to the category $\mathcal{V}ect_{\text{top}}(\mathbb{C})$ of topological $\mathbb{C}$-vector spaces.[2] Let us first describe $\mathsf{2Rie}$ in more detail.

For $\varepsilon > 0$ define the open annulus $A_\varepsilon = \{ p \in \mathbb{R}^2 \,|\, 1{-}\varepsilon < |p| < 1{+}\varepsilon \}$, as well as $A_\varepsilon^+ = \{ p \in A_\varepsilon \,|\, |p|{\geq}1 \}$ and $A_\varepsilon^- = \{ p \in A_\varepsilon \,|\, |p|{\leq}1 \}$. Denote by S^1 the unit circle. An object U of $\mathsf{2Rie}$ is a $k{+}1$-tupel $(\varepsilon; \Omega_1, \dots, \Omega_k)$, for some $k \geq 0$, where $\varepsilon > 0$ and each Ω_i is a smooth function from A_ε to $\mathbb{R}_{>0}$. Each of the Ω_i defines a metric $g(p) = \Omega_i(p)(dx^2 + dy^2)$ on A_ε. If we denote by $(A_\varepsilon)^{\sqcup k}$ the disjoint union of k copies of A_ε, then equally $(\varepsilon; \Omega_1, \dots, \Omega_k)$ defines a metric on $(A_\varepsilon)^{\sqcup k}$. This metric will be important to ensure that the composition of cobordisms via gluing leads again to a smooth metric.

A morphism $m : U \to V$ is an equivalence class of two-dimensional compact oriented Riemannian manifolds with parametrised boundaries, (M, ι, o). The equivalence relation will be given by parametrisation preserving isometries. To describe the parametrisation, suppose $U = (\varepsilon; \Omega_1, \dots, \Omega_k)$ and $V = (\nu; \sigma_1, \dots, \sigma_l)$. Then $\iota : (A_\varepsilon^+)^{\sqcup k} \to M$ and $o : (A_\varepsilon^-)^{\sqcup l} \to M$ are required to be orientation preserving isometries (onto their images) s.t. $\mathrm{Im}(\iota) \cap \mathrm{Im}(o) = \emptyset$ and $\partial M = \iota(S^1) \cup o(S^1)$. The special form of metric on A_ε is not a restriction because we can always choose isothermal coordinates.

[2] The target category is the main reason why what is presented here is only a "working definition". Topological vector spaces are needed to have a notion of continuity. However, it is likely that a more restricted class of topological vector spaces turns out to be the appropriate target category.

Composition of two morphisms $U \xrightarrow{m} V \xrightarrow{n} W$ in $\mathsf{2Rie}$ is again given by gluing with the maps ι and o. Specifically, $n \circ m$ is the equivalence class

$$n \circ m = \big[\,(G\,,\,\iota_M, o_N)\,\big] \qquad \text{where} \quad G = N \sqcup (A_\varepsilon)^{\sqcup l} \sqcup M/\sim \tag{2.13}$$

and the identification $\sim$ is given by

$$\iota_N(p) \sim p \;\text{ for }\; p \in (A_\varepsilon^+)^{\sqcup l} \qquad \text{and} \qquad o_M(p) \sim p \;\text{ for }\; p \in (A_\varepsilon^-)^{\sqcup l} \quad . \tag{2.14}$$

Since the composition always increases the area of a cobordism, just as was the case for $\mathsf{2Cpx}$, the category $\mathsf{2Rie}$ does not have unit morphisms and is hence a non-unital category.

A tensor product on $\mathsf{2Rie}$ is again given by disjoint union for morphisms (on objects, in $U \otimes V$ one takes the minimum value of ε in U, V). The trace is also defined as before. Given a morphism $m : U \otimes W \to V \otimes W$ with $m = \big[(M, \iota, o)\big]$, we set

$$\mathrm{tr}^{(W)}(m) = \big[\,\big(\,M \sqcup (A_\varepsilon)^{\sqcup l}/\sim\;,\;\iota_U, o_V\,\big)\,\big] \quad . \tag{2.15}$$

Here W is taken to be a $l{+}1$-tuple, and the equivalence relation is $p \sim \iota_W(p)$ for $p{\in}(A_\varepsilon^+)^{\sqcup l}$ and $p \sim o_W(p)$ for $p{\in}(A_\varepsilon^-)^{\sqcup l}$. (In fact, composition of morphisms is always a special case of the partial trace).

Remark 2.4:

(i) The parameter ε in the description of objects in $\mathsf{2Rie}$ can be removed by formulating everything in terms of function germs rather then via functions. We will however not do this here.

(ii) Intuitively, one may think of a functor $\mathsf{2Rie} \to \mathcal{V}ect_{\mathrm{top}}(\mathbb{C})$ as a continuum limit of a lattice model. Fix a two-dimensional surface Σ. A morphism in the lattice model is obtained by choosing a triangulation of that surface. One then passes to finer and finer triangulations of that same surface Σ. Taking each face to have the same area (keeping the overall area of Σ constant), one sees the appearance of a metric on Σ. It should be emphasised that there are few mathematical results about this continuum limit, and this view is supported mostly by the physical idea of renormalisation group flows and by computer simulations of statistical systems.

So far, we did not restrict how Z depends on the metric of the cobordisms. For a *conformal* field theory we demand a simple behaviour of Z if two different metrics are related by a Weyl transformation. Here is our working definition of a CFT.

A 2dCFT $(Z, \mathcal{H})$ of *central charge* $c \in \mathbb{C}$ obeys the following conditions.

(C1) $(Z, \mathcal{H})$ is a tensor functor [3] from $\mathsf{2Rie}$ to $\mathcal{V}ect_{\mathrm{top}}(\mathbb{C})$ which preserves the partial trace.

(C2) $\mathcal{H}$ is independent of the functions Ω_i, i.e. for $U = (\varepsilon, \Omega_1, \dots, \Omega_k)$ and $U' = (\varepsilon', \Omega_1', \dots, \Omega_k')$ we demand $\mathcal{H}(U) = \mathcal{H}(U')$.

(C3) Let U, U' be two $k{+}1$ tupels and V, V' be two $l{+}1$ tupels. Then by (ii), $\mathcal{H}(U) = \mathcal{H}(U')$

[3] Here is another point where we are being negligent: the proper definition of the tensor product to be used in the target category. These functional analytic aspects will not be treated (nor needed) in this text. A requirement for the appropriate definition of the target category is the existence of a well-behaved tensor product.

and $\mathcal{H}(V) = \mathcal{H}(V')$. Consider two morphisms $m : U \to V$ and $m' : U' \to V'$ represented by manifolds M and M' with metrics g, g', respectively, and parametrised boundaries. Suppose there is an orientation preserving conformal transformation $f : M \to M'$ which is compatible with the boundary parametrisation, so that in particular $(f^* g')(p) = e^{2\sigma(p)} g(p)$ for some function $\sigma : M \to \mathbb{R}$. Then

$$Z(m) = e^{c S_{\text{liou}}(\sigma)} Z(m') \quad , \tag{2.16}$$

where $S[\sigma] \in \mathbb{C}$ is the (suitably normalised) Liouville action, see e.g. [Ga2] for details.

Remark 2.5:
(i) In physical terms, property (C2) means that in a 2dCFT the space of states does not change under local scale transformations of the boundary circle. This will lead to the state-field correspondence.

(ii) Property (C3) implies in particular, that if two metrics on a given manifold are related by a Weyl-transformation, then the corresponding state sums Z differ only by a scalar factor. In other words, a 2dCFT is covariant under local scale transformations.

(iii) A 2dTFT as defined in section 2.1 is an example of a 2dCFT. In this case Z is altogether independent of the metric g. Compatibility with (C3) then requires the central charge c to be zero.

(iv) Even with the restricted dependence on the metric of the Riemannian manifold as required in (C3), a 2dCFT remains an extremely complicated object. The construction of $(Z, \mathcal{H})$ is basically only known for the topological case and for certain free field theories, like free bosons where the properties of Z on a given Riemannian manifold are related to the Laplace operator, see e.g. [Se1, HK2].

Compact CFT

As a first simplification, we restrict ourselves to what might be called "compact CFTs". Denote by S_r the object $(\varepsilon, \Omega \equiv r^{-2})$ of $\mathsf{2Rie}$, i.e. A_ε carries the standard metric of $\mathbb{R}^2$, multiplied by a constant factor r^{-2}. Consider the annulus $A(R,r) = \{\, p \in \mathbb{R}^2 \,|\, r<|p|<R \,\}$ with metric given by the restriction of the standard metric on $\mathbb{R}^2$. This can be turned into a morphism $A(R,r) : S_r \to S_R$ of $\mathsf{2Rie}$ by choosing the parametrisations $\iota(p) = r \cdot p$ and $o(p) = R \cdot p$.

Denote $\mathcal{H}(S_r) = \mathcal{H}(S_R) \equiv \mathsf{H}$. Then from $A(R,r)$ we get a linear map $Z(A(R,r)) : \mathsf{H} \to \mathsf{H}$. We would like to think of $Z(A(R,r))$ as in some sense "being close to the identity operator", because for $r \approx R$, $A(r,R)$ is "close to the (non-existing) unit morphism in $\mathsf{2Rie}$". Concretely we demand $Z(A(R,r))$ to have the following nice properties.

- By (C3), $Z(A(R,r))$ only depends on the ratio $q = r/R$. Using this we define $U(r/R) = Z(A(R,r))$ for any choice of $R > r$. This is called the *dilation operator*, cf. [DMS, section 6.2]. From functoriality of Z it follows that $U(q_1)U(q_2) = U(q_1 q_2)$. We demand that there is an operator $D : \mathsf{H} \to \mathsf{H}$ with discrete, real spectrum S bounded from below, s.t. $U(q) = q^D$.
- Let $\mathcal{F} = \oplus_{\Delta \in \mathsf{S}} V_\Delta$ be the direct sum of eigenspaces $\mathcal{F}_\Delta$ of D with eigenvalue Δ. We demand that $\dim \mathcal{F}_\Delta < \infty$ and that $\mathcal{F}$ is dense in H. The eigenvalues Δ are called *scaling dimensions.*

Remark 2.6:
The name "compact" as well as the properties listed above are again motivated by the physical picture. The operator D is closely related to the Hamiltonian of the CFT on a cylinder.

The properties of D imply that the Hamiltonian has discrete spectrum and finite degeneracy of each energy level. The discreetness of the spectrum is typical for a system with compact configuration space. On the mathematical side an analogue is provided by the Laplace or Dirac operator, which have a discrete spectrum on compact manifolds.

2.5 Correlation functions

In chapters 4 and 6, we will mainly use the language of correlation functions to describe a 2dCFT. Let us have a short look at how this is related to the functorial formulation.

Field insertions

In quantum field theory one considers correlation functions of fields inserted at certain points (or of Wilson loops, in gauge theories), rather than a functor $(Z, \mathcal{H})$. For a compact CFT, these two pictures are related by the state-field correspondence, which will be outlined below.

By a (closed, oriented) *Riemannian world sheet* X^g we mean an oriented, compact two dimensional Riemannian manifold with empty boundary, also denoted by X^g, with a finite, ordered set of distinct marked points $p_1, \dots, p_n$. For each marked point p_k there is a germ $[f_k]$ of orientation preserving local isometries from a disc shaped neighbourhood of zero $D_\varepsilon = \big\{ p \in \mathbb{R}^2 \,\big|\, |p| < \varepsilon \big\}$ to X^g s.t $f_k(0) = p_k$. The notation X^g is to remind of the presence of the metric; in chapter 6 a topological world sheet X will be used, which does not carry a metric. However, until chapter 6 we will only be dealing with Riemannian world sheets, which will be called "world sheets" for short.

Given a compact 2dCFT $(Z, \mathcal{H})$, we would like to construct an assignment

$$\mathrm{X}^g \longmapsto C(\mathrm{X}^g) \qquad \text{where} \quad C(\mathrm{X}^g) : \underbrace{\mathcal{F} \otimes \cdots \otimes \mathcal{F}}_{\#(\text{field ins.})\text{ copies}} \longrightarrow \mathbb{C} \quad . \tag{2.17}$$

The linear functional $C(\mathrm{X}^g)$ is called *correlation function* of the 2dCFT on the world sheet X^g. To obtain the correlation functions, one first constructs a morphism $\mathrm{X}^g_\varepsilon : U \longrightarrow \emptyset$ in $\mathsf{2Rie}$ from the world sheet X^g and then defines $C(\mathrm{X}^g)$ in terms of $Z(\mathrm{X}^g_\varepsilon)$.

In more detail, let ε be small enough s.t for each k, at the the k'th marked point we can choose a representative $f_k : D_\varepsilon \to \mathrm{X}^g$ of the coordinate germ $[f_k]$. Define the map $\iota_k : A^+_\varepsilon \to \mathrm{X}^g$ to be $\iota_k(p) = f_k(\varepsilon p/2)$. Let n be the number of marked points on X^g and consider the object $U = \big(\varepsilon;\, \Omega_1 \equiv \varepsilon^2/4\,, \cdots\,,\, \Omega_n \equiv \varepsilon^2/4\big)$ of $\mathsf{2Rie}$. One can verify that taking the manifold X^g_ε obtained by cutting the image of the disc $D_{\varepsilon/2}$ under each of the f_k out of X^g, together with the parametrisation $\iota : (A^+_\varepsilon)^{\sqcup n} \to \mathrm{X}^g$ given by the union of the ι_k, is a morphism

$$\mathrm{X}^g_\varepsilon : U \longrightarrow \emptyset \tag{2.18}$$

in $\mathsf{2Rie}$ (more precisely, the morphism is given by the equivalence class of X^g_ε, which will also be denoted by X^g_ε). By definition, $Z(\mathrm{X}^g_\varepsilon)$ is a linear map $\mathsf{H}^{\otimes n} \to \mathbb{C}$. We define $C(\mathrm{X}^g)$ by prescribing its values on tupels $(v_1, \dots, v_n)$ where each v_k has a definite scaling dimension $v_k \in \mathcal{F}_{\Delta_k}$,

$$C(\mathrm{X}^g)(v_1, \dots, v_n) = \varepsilon^{-\Delta_1 - \cdots - \Delta_n} Z(\mathrm{X}^g_\varepsilon)(v_1, \dots, v_n) \quad . \tag{2.19}$$

For a different choice ε' instead of ε, the definitions can be related by gluing appropriate annuli $A(\varepsilon/2, \varepsilon'/2)$ to X^g_ε. The simple form of $Z(A(\varepsilon/2, \varepsilon'/2))$ for compact CFTs then ensures that the definition (2.19) is actually independent of ε.

Conversely, if we are given H and the assignment C, we can recover the functor $(Z,\mathcal{H})$. To see this, first note that by property (C1), $\mathcal{H}(U)$ is fixed to be $\mathsf{H}^{\otimes k}$ if U has k components. Second, it is enough to give $Z(M)$ for cobordisms of the form $M : U \to \emptyset$. For a general cobordism $N : U \to V$, we can glue cylinders with two ingoing boundaries to all connected components of V. Each cylinder defines a non-degenerate pairing $\mathsf{H}\times\mathsf{H} \to \mathbb{C}$ and thus allows to recover $Z(N)$ starting only from Z restricted to morphisms with target $\emptyset$. Third, by property (C3) above, it is enough to give $Z(M)$ for morphisms $M : U \to \emptyset$ where U is of the form $U = (\varepsilon, \Omega_1 \equiv 1, \dots, \Omega_k \equiv 1)$. In this case we simply set, for $v_1, \dots, v_n \in \mathcal{F}$,

$$Z(M)(v_1,\dots,v_n) = C(\mathrm{X}^g)(v_1,\dots,v_n) \quad , \tag{2.20}$$

where X^g is the world sheet obtained by gluing unit discs into the holes of M using the map ι. As marked points on X^g we take the centres of the unit disc, with the identity function as local coordinate. It is not difficult to check that this construction is inverse to the construction of C in terms of Z.

What have we gained by doing this? We see that the data of the functor $(Z,\mathcal{H})$ can equivalently be encoded in the pair (H, C). In quantum field theory, one works mostly with correlation functions, i.e. in the formulation (H, C). In fact, both formulations have their merit. In the functorial language, the consistency conditions are easiest to formulate. The language of correlation functions is very powerful for concrete calculations, since in many cases the correlators obey differential equations acting on the positions of the field insertions.

Factorisation of correlators

Factorisation is a consistency conditions for correlators, which in the functorial formulation corresponds to compatibility of Z with composition and the partial trace.

Let $m = \big[\,(M,\iota,o)\,\big]$ be a morphism from U to V. Consider the annulus A_ε with metric $g(p) = \Omega(p)\big(dx^2{+}dy^2\big)$. Given an isometric embedding $f : A_\varepsilon \to M$, we can construct a new morphism $\Lambda_f(m) : U\otimes W \to V \otimes W$, for $W = (\varepsilon,\Omega)$, by "cutting M along the image of S^1 under f". That is, let $M' = M\backslash f(S^1)$ be given by M minus the image of the unit circle under f. Then

$$\Lambda_f(m) = \big[\,\big(\, M' \sqcup A_\varepsilon^+ \sqcup A_\varepsilon^- / \sim \;\; , \;\; \iota, o \big)\,\big] \quad , \tag{2.21}$$

where the identification $\sim$ is given by $f(p) \sim p$ for $p \in A_\varepsilon^+$ and equally $f(p)\sim p$ for $p \in A_\varepsilon^-$. Thus $\Lambda_f(m)$ has one ingoing and one outgoing boundary component more than m, parametrised by f restricted to A_ε^+ and A_ε^-, respectively.

Note that taking the partial trace is left-inverse to this procedure of "cutting along an S^1", $\mathrm{tr}^{\,(W)}\big(\Lambda_f(m)\big) = m$. Applying Z to both sides yields

$$\mathrm{tr}^{\,(H)} Z(\Lambda_f(m)) = Z(m) \quad , \tag{2.22}$$

which in physical terms is nothing by the sum over intermediate states.

The corresponding identity for correlators is called *factorisation* and takes the form

$$\sum_{\alpha,\beta} U_{\alpha,\beta}\, C\big(\Gamma_f(\mathrm{X}^g)\big)(v_1,\dots,v_n,u_\alpha,u_\beta) = C\big(\mathrm{X}^g\big)(v_1,\dots,v_n) \quad . \tag{2.23}$$

This follows when choosing $m = \mathrm{X}^g_\varepsilon$ in (2.22), with X^g_ε as given in (2.18). To turn the lhs of (2.22) back into a correlator, one has to change the outgoing boundary of $\Lambda_f(\mathrm{X}^g_\varepsilon)$ to an ingoing boundary by gluing an annulus with two ingoing boundaries. All ingoing holes are then removed by gluing discs with fields inserted in their centres, as above. This results in a world sheet with two more field insertions as X^g, and which as been denoted by $\Gamma_f(\mathrm{X}^g)$ on the lhs of (2.23). The matrix $U_{\alpha,\beta}$ compensates for the effect of gluing the annulus, i.e. it is related to the inverse of the two-point correlator on a sphere.

In fact, when choosing X^g to be a sphere with two field insertions, then cutting X^g along an S^1 produces two spheres with two field insertions. The correlator of one of these cancels against the matrix $U_{\alpha,\beta}$.

Holomorphic fields

A holomorphic field of weight Δ is an element W of $\mathcal{F}_\Delta$ with the property that all correlators with an insertion of W depend holomorphically on the insertion point.

Concretely, let X^g be a world sheet and p be one of the marked points, with coordinate germ $[f]$. Choose a representative $f : D_\varepsilon \to \mathrm{X}^g$. We can then define a new world sheet $\mathrm{X}^g(z)$ to be equal to X^g except for the marked point p, which gets replaced by $\tilde{p} = f(z)$, with local coordinates $\tilde{f} : D_\delta \to \mathrm{X}^g(z)$, $\tilde{f}(\zeta) = f(\zeta + z)$. This is well defined for $|z| < \varepsilon$ and δ small enough. Suppose p is the first of n marked points of X^g. Choose vectors $v_2, \dots, v_n \in \mathsf{H}$. An element $W \in \mathcal{F}_\Delta$ is a *holomorphic field* of weight Δ if

$$\frac{d}{dz} C\big(\mathrm{X}^g(z)\big)(W, v_2, \dots, v_n) = 0 \tag{2.24}$$

for all choices of v_k and for all world sheets X^g. This is an infinite set of conditions. However, using factorisation we can always cut the world sheet X^g along an S^1 containing only W and no other field insertion. It is then enough to know that (2.24) holds for all two-point functions on the sphere (with one W and one other insertion).

In the same way, a field $\overline{W} \in \mathcal{F}_\Delta$ is an *anti-holomorphic field* of weight Δ if

$$\frac{d}{d\bar{z}} C\big(\mathrm{X}^g(z)\big)(\overline{W}, \phi_2, \dots, \phi_n) = 0 \tag{2.25}$$

for all choices of v_k and for all world sheets X^g.

Remark 2.7 :
(i) Holomorphic and anti-holomorphic fields should be thought of as symmetries of the CFT. Via contour integration they generate an infinite set of relations between correlation functions, the so-called *Ward identities*. It is beyond the scope of this introduction to explain this in any detail, see e.g. [DMS, section 5.2] for more information. Nonetheless it should at least be mentioned that the most important holomorphic field is the stress tensor $T \in \mathcal{F}_2$, which also has an anti-holomorphic partner $\bar{T} \in \mathcal{F}_2$. The corresponding symmetry is the covariance of correlation functions under Weyl transformations of the metric, see [Ga2, lecture 2] where this point is further developed. The stress tensor is also responsible for the appearance of the *Virasoro algebra* in conformal field theory, see [DMS, section 6.2] for an introduction from the physics point of view.

(ii) If we restrict ourselves to correlators of holomorphic fields on the complex plane, we obtain a so-called *meromorphic conformal field theory*. For such theories there exist an axiomatic formulation [Go, GG]. This is also the motivation for the introduction of *conformal vertex algebras*, a point to which we will return in chapter 4.

2.6 Surfaces with boundaries and unoriented surfaces

In the beginning of the previous section we introduced oriented, closed Riemannian world sheets X^g. Let us first extend this notion to *oriented Riemannian world sheets* by allowing the surface X^g to have non-empty boundary. In this case we first need to fix a set $\mathcal{B}$, the set of boundary conditions. In addition to the marked points p_k in the interior $\mathrm{X}^g\backslash\partial\mathrm{X}^g$ of the world sheet, there is a finite, ordered set of distinct marked points $\{q_1,\dots,q_m\}$ on the boundary $\partial\mathrm{X}^g$. For each marked point $q_l\in\partial\mathrm{X}^g$ there is a germ $[g_l]$ of orientation preserving local isometries from a half-disc shaped neighbourhood of zero $H_\varepsilon=\big\{\,p{\equiv}(p_1,p_2)\in\mathbb{R}^2\,\big|\,|p|<\varepsilon\,,\;p_2{\geq}0\big\}$ to X^g such that $g_l(0)=q_l$ and the interval $]-\varepsilon,\varepsilon[$ on the x-axis gets mapped to $\partial\mathrm{X}^g$. Finally, each segment of $\partial\mathrm{X}^g\backslash\{q_1,\dots,q_m\}$ gets assigned a boundary condition, i.e. it gets labelled by an element of $\mathcal{B}$.

A CFT that is defined on oriented world sheets requires more structure than a CFT only defined on oriented closed world sheets. Let us call the former an open/closed oriented CFT and the latter a closed oriented CFT. In particular, an open/closed oriented CFT always gives rise to a closed oriented CFT by simply restricting to world sheets with empty boundary. However, not every closed oriented CFT can arise in this way.

The additional structure we need is first, the set of boundary conditions $\mathcal{B}$ already mentioned above, and second, for each pair $a,b\in\mathcal{B}$ a $\mathbb{C}$-vector space $\mathcal{F}_{ab}$, the spaces of boundary fields. The CFT is again defined by an assignment of correlators $\mathrm{X}^g\longmapsto C(\mathrm{X}^g)$, but as opposed to (2.17) we now have to take into account the marked boundary points

$$C(X):\mathcal{F}_{a_1b_1}\otimes\mathcal{F}_{a_2b_2}\cdots\otimes\mathcal{F}\otimes\mathcal{F}\cdots\longrightarrow\mathbb{C}\quad, \tag{2.26}$$

where a_l and b_l refer to the label assigned to the boundary segment to the left and to the right of the insertion point q_l, respectively (the boundary $\partial\mathrm{X}^g$ is oriented by the orientation of X^g).

The boundary conditions $\mathcal{B}$ have to be *conformal* in the following sense. There is an embedding of the subset of holomorphic and anti-holomorphic fields of $\mathcal{F}$ into each $\mathcal{F}_{aa}$. For the holomorphic and anti-holomorphic component T and $\bar T$ of the stress tensor (cf. Remark 2.7 (i)), we require that for any world sheet X^g with at least one boundary insertion, $C(\mathrm{X}^g)(T,\dots)=C(\mathrm{X}^g)(\bar T,\dots)$, where $C(\mathrm{X}^g):\mathcal{F}_{aa}\otimes\cdots\to\mathbb{C}$ (see [C1, C3] for the physical reasoning behind this). In fact, in section 4.2 below we will require a similar identity to hold for a larger set of holomorphic and anti-holomorphic fields. In physical terms, we then consider boundary conditions which preserve more than just conformal symmetry.

The consistency conditions for an open/closed CFT take a more complicated form than those discussed in section 2.4 and 2.5 for a closed CFT. Again, these conditions can be expressed in the functorial formulation of the theory. For this one needs to consider a different cobordism category, where in addition to the annuli making up the objects of $\mathsf{2Rie}$, there are also rectangles $[-1,1]{\times}[-\varepsilon,\varepsilon]$, endowed with a metric. Further, the cobordisms are then Riemannian manifolds with corners, see also [HK2]. We will not develop any further details of this approach here. In the algebraic setting, the consistency conditions are formulated in terms of correlators in Problem 6.6 below.

Finally, one can also consider CFTs defined on *Riemannian world sheets*. These are defined in the same way as oriented Riemannian world sheets X^g, except that one does not specify an orientation on X^g (and thus also does not require the local coordinates around the marked points to be orientation preserving). Instead one has to fix an orientation of the boundary $\partial\mathrm{X}^g$. Let us denote a CFT defined on closed Riemannian world sheets as an unoriented closed CFT, and a CFT defined on Riemannian world sheets as an unoriented open/closed CFT. A closed or open/closed oriented CFT can be obtained from an unoriented closed or open/closed CFT by restricting to oriented world sheets, but again one does not obtain every oriented CFT in this way.

The algebraic construction of CFT correlators in section 6 is given for oriented open/closed CFTs. The unoriented open/closed case will also be mentioned, but for the details the reader will be referred to [II] and [IV].

3 Three-dimensional topological field theory

Three dimensional topological field theory (3dTFT) is a well developed mathematical machine to construct invariants of three manifolds with embedded ribbon graphs from a modular tensor category. It was originally developed by Reshetikhin and Turaev [RT1, RT2, Tu1, Tu] and is reviewed e.g. in [BK,KRT] as well as in section I:2. In the present chapter only a short overview will be given.

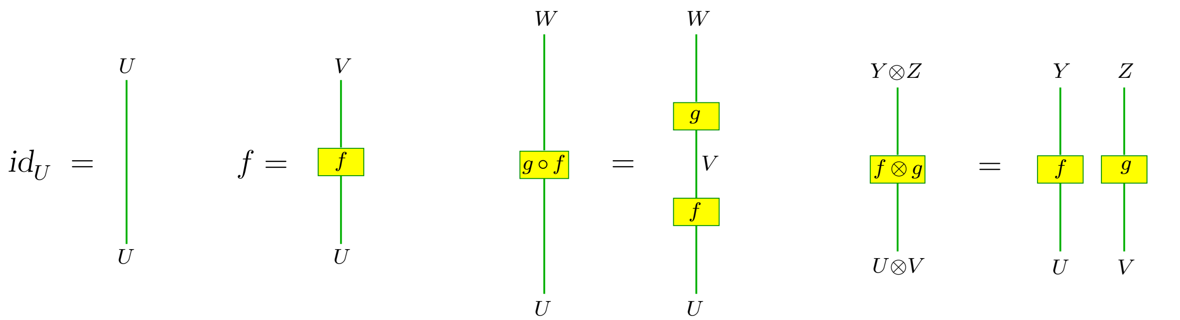
3.1 Ribbon categories

As a convention, in this text we take all categories to be small. To define a modular tensor category let us first recall the notion of a ribbon category [JS1, JS2], see [Ks, chapter XIV] for a more thorough introduction.

A *ribbon category* $\mathcal{C}$ is a tensor category with the following additional structure. To every object $U \in \mathrm{Obj}(\mathcal{C})$ one assigns an object $U^\vee \in \mathrm{Obj}(\mathcal{C})$, called the (right-) dual of U, and there are three families of morphisms,

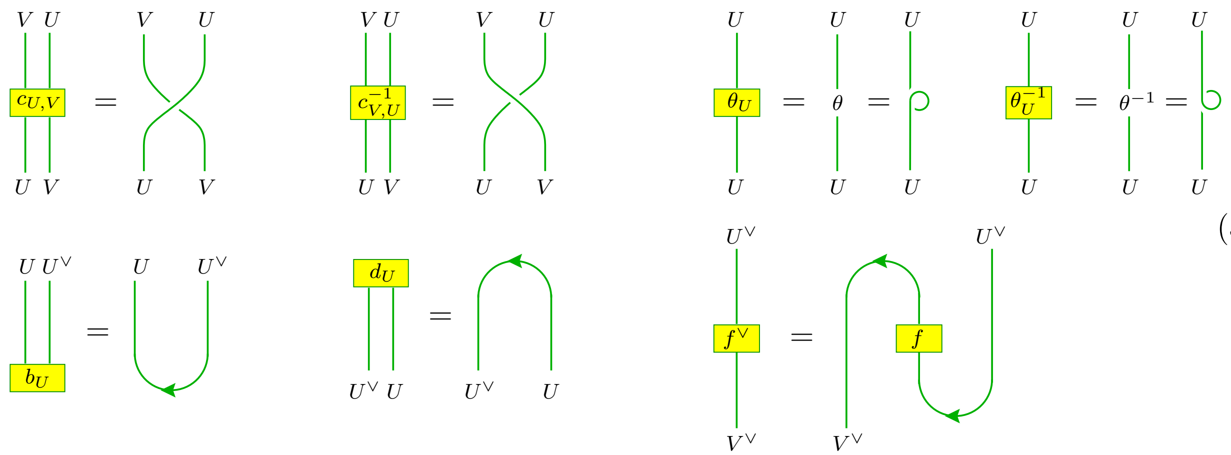
$$\begin{aligned} &\text{(Right-) Duality:} \quad && b_U \in \mathrm{Hom}(\mathbf{1}, U \otimes U^\vee)\,, \quad d_U \in \mathrm{Hom}(U^\vee \otimes U, \mathbf{1})\,, \\ &\text{Braiding :} && c_{U,V} \in \mathrm{Hom}(U \otimes V, V \otimes U)\,, \\ &\text{Twist :} && \theta_U \in \mathrm{Hom}(U, U) \end{aligned} \tag{3.1}$$

for all $U \in \mathrm{Obj}(\mathcal{C})$, respectively for all $U, V \in \mathrm{Obj}(\mathcal{C})$, subject to certain compatibility conditions (see definition C:2.1). Instead of detailing these conditions, let us introduce a graphical representation of the morphisms via ribbons.



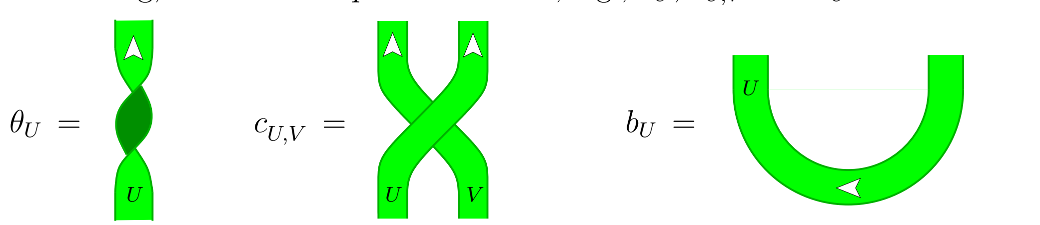
(3.2)

In these pictures, the lines stand for ribbons that lie flat in the paper plane. The surface of a ribbon is oriented and we will refer to this orientation as "white" and "black" side of a ribbon; the ribbons implied by the lines in (3.2) face the reader with their white side. This abbreviation is referred to as blackboard-framing. The morphisms in (3.1) are represented as



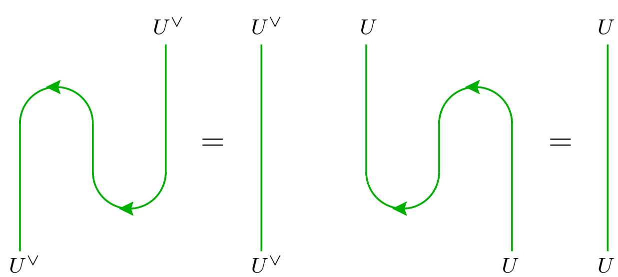
(3.3)

Ribbons labelled by the tensor unit **1** are not drawn in the graphical notation. If one does not use backboard-framing, the ribbon representation of, e.g., θ_U, $c_{U,V}$ and b_U looks as



(3.4)

The compatibility conditions on the morphisms (3.1) amount to the statement that deformations of the ribbons in the ribbon-representation of a morphism do not change the corresponding morphism in $\mathcal{C}$. For example,



(3.5)

In terms of the morphisms (3.1), the first of these identities amounts to $(d_U \otimes id_{U^\vee}) \circ (id_{U^\vee} \otimes b_U) = id_{U^\vee}$ which holds by definition of the right-duality. In a ribbon category there is automatically also a left-duality $\tilde{b}_U$, $\tilde{d}_U$ which coincides with the right-duality on objects and where

$$\tilde{b}_U \in \mathrm{Hom}(\mathbf{1}, U^\vee \otimes U) \ , \quad \tilde{d}_U \in \mathrm{Hom}(U \otimes U^\vee, \mathbf{1}) \ , \tag{3.6}$$

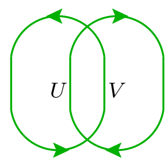
see e.g. [Ks, Proposition XIV.3.5] and figure (I:2.12). One also defines the *trace* of an endomorphism $f \in \mathrm{Hom}(U,U)$ as

$$\mathrm{tr}(f) := d_U \circ (id_{U^\vee} \otimes f) \circ \tilde{b}_U = \tilde{d}_U \circ (f \otimes id_{U^\vee}) \circ b_U \,. \tag{3.7}$$

The two expressions for $\mathrm{tr}(f)$ can be verified to coincide by definition of the dualities. The *quantum dimension* of an object U is defined as

$$\dim(U) := \mathrm{tr}(id_U) \quad . \tag{3.8}$$

The important feature of ribbon categories is that they give homotopy invariants of ribbon graphs in S^3 [RT1], see [Tu, chapter I] or [BK, chapter 2] for a review. For example, the graph

$s_{U,V} \ :=$ 

(3.9)

gives an element of $\mathrm{Hom}(\mathbf{1},\mathbf{1})$. Expressed in terms of the morphisms (3.1), (3.6) it reads

$$s_{U,V} = (d_V \otimes \tilde{d}_U) \circ [\, id_{V^\vee} \otimes (c_{U,V} \circ c_{V,U}) \otimes id_{U^\vee}] \circ (\tilde{b}_V \otimes b_U) \tag{3.10}$$

which is nothing by the trace of the endomorphism $c_{U,V} \circ c_{V,U}$ of $V \otimes U$.

3.2 Modular tensor categories

Given a category $\mathcal{C}$, let us call the set of isomorphism classes of simple objects of $\mathcal{C}$ the *index set* $\mathcal{I}$. Let $\Bbbk$ be a field. A *modular tensor category* [Tu1] is a strict $\Bbbk$-linear abelian semisimple ribbon category, s.t. the index set $\mathcal{I}$ is finite, every simple object is absolutely simple, and for which the *s-matrix* $s = (s_{i,j})_{i,j \in \mathcal{I}}$ with entries

$$s_{i,j} := s_{U_i,U_j} = \mathrm{tr}(c_{U_i,U_j} \circ c_{U_j,U_i}) \tag{3.11}$$

is non-degenerate. An element $\mathcal{D} \in \Bbbk$ is called *rank* of a modular tensor category $\mathcal{C}$ if

$$\mathcal{D}^2 = \sum_{i \in \mathcal{I}} (\dim\, U_i)^2 \quad . \tag{3.12}$$

Given a simple object U_i of $\mathcal{C}$, also $U_i^\vee$ is simple, and thus $U_i^\vee \cong U_{\bar\imath}$ for some $\bar\imath \in \mathcal{I}$. The assignment $i \mapsto \bar\imath$ defines an involution on $\mathcal{I}$.

Remark 3.1:

(i) Recall that an object V of an abelian category is *simple* iff any injection $U \hookrightarrow V$ is either zero or an isomorphism. An object V of a $\Bbbk$-linear abelian category is called *absolutely simple* iff $\mathrm{Hom}(V,V) = \Bbbk\, id_V$. If $\mathcal{C}$ is semisimple and $\Bbbk$ is algebraically closed, absolutely simple is equivalent to simple.

(ii) In the original definition of a modular tensor category (see [Tu1] and [Tu, section II.1.4]), $\Bbbk$ is replaced by a commutative ring, semisimplicity is replaced by the weaker dominance property, and abelian by the weaker property additive.

(iii) The restriction to strict categories in the definition of modular is done merely for convenience. If a ribbon category is not strict, we can always replace it by an equivalent strict category via MacLane's coherence theorem (cf. [ML, section XI.3] or [Ks, section XI.5]). This must in particular be done for some of the examples listed below.

(iv) The existence (and choice) of a rank $\mathcal{D} \in \Bbbk$ is required in the construction of a 3dTFT from a modular tensor category, see [Tu, section 1.6] and section 3.4 below.

The simplest example of a modular tensor category is the category $\mathcal{V}ect_f(\Bbbk)$ of finite dimensional $\Bbbk$-vector spaces. On the other hand, the category of representations of a finite group is ribbon, but in general not modular, since the braiding is symmetric and hence $s_{i,j} = \dim(U_i)\dim(U_j)$ is degenerate. An example for a category of representation that is modular is provided by integrable representations of a semi-simple affine Lie algebra at positive integer level. Recently, quite a few results have been obtained that characterise cases when certain representation categories are modular:

- If H is a connected C^* weak Hopf algebra, then the category of unitary representations of its double is a unitary modular tensor category [NTV].
- Similarly, the representation category of a connected ribbon factorisable weak Hopf algebra over $\mathbb{C}$ (or, more generally, over any algebraically closed field $\Bbbk$) with a Haar integral is modular [NTV].
- If a finite-index net of von Neumann algebras on the real line is strongly additive (which for conformal nets is equivalent to Haag duality) and has the split property, its category of local sectors is a modular tensor category [KLM].
- Finally, according to the results of [Hu1], if a self-dual vertex algebra that obeys Zhu's C_2 cofiniteness condition and certain conditions on its homogeneous subspaces has a semi-simple representation category, then this category is actually a modular tensor category.

Another class of examples for modular tensor categories is provided by theta-categories.

3.3 Example: Theta-categories

An object V of a tensor category $\mathcal{C}$ is called *invertible* iff there exists an object V' such that $V \otimes V'$ is isomorphic to the tensor unit $\mathbf{1}$. A *theta-category* [FK] is a $\Bbbk$-linear abelian semisimple ribbon category in which every simple object is invertible.

To obtain examples of theta-categories, choose a finite abelian group G and consider the category of finite-dimensional G-graded $\Bbbk$-vector spaces with the grade-respecting linear maps as morphisms. This is an abelian semisimple category whose isomorphism classes of simple objects L_g (a one-dimensional vector space of grade g) are in bijection to the elements g of G. A general object V can be written as a direct sum $\bigoplus_{g\in G} V_g$ of finite-dimensional vector spaces. Every three-cocycle $\psi \in Z^3(G, \Bbbk^\times)$ defines an associator via

$$\begin{array}{rrcl} \alpha_{V,V',V''}: & \big(V_{g_1} \otimes V'_{g_2}\big) \otimes V''_{g_3} & \rightarrow & V_{g_1} \otimes \big(V'_{g_2} \otimes V''_{g_3}\big) \\ & (v \otimes v') \otimes v'' & \mapsto & \psi(g_1, g_2, g_3)^{-1}\, v \otimes (v' \otimes v'') \,. \end{array} \tag{3.13}$$

A short review of group cohomology can be found in appendix III:A. The pentagon identity for $\alpha_{V,V',V''}$ translates into the cocycle condition for $\psi(g_1, g_2, g_3)$. The triangle identity for the unit constraint is trivially fulfilled because we simply identify $\Bbbk \otimes_\Bbbk V = V = V \otimes_\Bbbk \Bbbk$. It turns out ([JS2], see also [FK]) that a braiding and a twist can be obtained from a representative (ψ, Ω) of the third abelian group cohomology of G (as defined in [EM] and summarised in appendix III:A). We define

$$\begin{array}{rrcl} c_{V,V'}: & V_{g_1} \otimes V'_{g_2} & \rightarrow & V'_{g_2} \otimes V_{g_1} \\ & v \otimes v' & \mapsto & \Omega(g_2, g_1)^{-1}\, v' \otimes v \end{array} \qquad \begin{array}{rrcl} \theta_V: & V_g & \rightarrow & V_g \\ & v & \mapsto & \Omega(g,g)^{-1}\, v \end{array} \;. \tag{3.14}$$

In this way one obtains a ribbon category, which we will denote by $\mathcal{C}(G, \psi, \Omega)$. One can also show (see e.g. Proposition III:2.11) that every theta-category with finite index set $\mathcal{I}$ is equivalent to an appropriate $\mathcal{C}(G, \psi, \Omega)$. More details on theta-categories can be found in section III:2.

For $\mathcal{C}(G, \psi, \Omega)$ to be also modular, the only condition that remains to be verified is non-degeneracy of the s-matrix. The latter is easily calculated to be

$$s_{g,h} = \big(\Omega(g,h)\Omega(h,g)\big)^{-1} \quad . \tag{3.15}$$

Finally, since we defined a modular tensor category to be strict, while $\mathcal{C}(G, \psi, \Omega)$ is not (for $\psi \neq 1$), we also have to invoke MacLane's coherence theorem to replace $\mathcal{C}(G, \psi, \Omega)$ by an equivalent strict ribbon category $\mathcal{C}^{\rm str}(G, \psi, \Omega)$.

3.4 3dTFT from modular tensor categories

A 3dTFT will again be formulated as a functor. As opposed to the 2dTFT in section 2.1, here the cobordisms are three-manifolds and the objects two-manifolds. In fact, one can define [Wt1,At1] d dimensional topological field theories with d-dimensional cobordisms as morphisms and $d{-}1$ dimensional manifolds as objects, for reviews see [Q], [BK, section 4.2] or [Tu, chapter III].

Let $\mathcal{C}$ be a modular tensor category. The category $\mathsf{3Cob}(\mathcal{C})$ has extended surfaces as objects and weighted cobordisms between extended surfaces as morphisms. An *extended surface* E consists of the following data:

- A compact oriented two-dimensional manifold with empty boundary, also denoted by E.
- A finite (unordered) set of marked points – that is, of quadruples $(p_i, [\gamma_i], V_i, \varepsilon_i)$, where the $p_i \in E$ are mutually distinct points of the surface E and $[\gamma_i]$ is a germ of arcs [4] γ_i: $[-\delta, \delta] \rightarrow E$ with $\gamma_i(0) = p_i$. Furthermore, $V_i \in \mathcal{O}bj(\mathcal{C})$, and $\varepsilon_i \in \{\pm 1\}$ is a sign.

[4] By a *germ of arcs* we mean an equivalence class $[\gamma]$ of continuous embeddings γ of intervals $[-\delta, \delta] \subset \mathbb{R}$ into the extended surface E. Two embeddings γ: $[-\delta, \delta] \rightarrow E$ and γ': $[-\delta', \delta'] \rightarrow E$ are equivalent if there is a positive $\varepsilon < \delta, \delta'$ such that γ and γ' are equal when restricted to the interval $[-\varepsilon, \varepsilon]$.

- A Lagrangian subspace $\lambda \subset H_1(E,\mathbb{R})$.

A morphism $E \to E'$ is a *weighted cobordism* M, consisting of the following data[5]:

- A compact oriented three-manifold, also denoted by M, such that $\partial\mathrm{M} = (-E) \sqcup E'$.
  Here $-E$ is obtained from E by reversing its 2-orientation and replacing any marked point $(p,[\gamma],U,\varepsilon)$ by $(p,[\tilde\gamma],U,-\varepsilon)$ with $\tilde\gamma(t) = \gamma(-t)$. The boundary $\partial\mathrm{M}$ of a cobordism is oriented according to the inward pointing normal.

- A ribbon graph R inside M such that for each marked point $(p,[\gamma],U,\varepsilon)$ of $(-E)\sqcup E'$ there is a ribbon ending on $(-E)\sqcup E'$. The notion of a ribbon graph is reviewed in section I:2.3 and the allowed ways for a ribbon to end on an arc are shown explicitly in (IV:3.1).

- An integer m, called the *weight*. If $\mathrm{M} = \emptyset$ then we require $m = 0$.

To make explicit the weight of a morphism in $\mathsf{3Cob}(\mathcal{C})$ we will write (M,m) instead of M. In the composition $(N,n) = (\mathrm{M}',m') \circ (\mathrm{M},m)$ of two morphisms $(\mathrm{M},m) : E \to E'$ and $(\mathrm{M}',m') : E' \to E''$, the cobordism N is obtained by identifying M and M$'$ along E' and the weight n is computed from m, m' and the Lagrangian subspaces of E, E' and E''. Details can be found in [Tu, section IV.9.1]. The tensor product in $\mathsf{3Cob}(\mathcal{C})$ is given on objects by disjoint union $E \otimes E' = E \sqcup E'$ and on morphisms by $(M,m) \otimes (M',m') = (M \sqcup M', m+m')$.

Given the modular tensor category $\mathcal{C}$ with rank $\mathcal{D}$ (so that in particular, a rank exists), one can construct a tensor functor $(Z,\mathcal{H}) : \mathsf{3Cob}(\mathcal{C}) \to \mathcal{V}ect_f(\Bbbk)$ [Tu, Theorem IV.9.2.1]. Here again, $\mathcal{H}$ denotes the action of the functor on objects, i.e. $\mathcal{H}(E)$ is a finite-dimensional $\Bbbk$-vector space, and Z denotes the action on morphisms, s.t. $Z(E \xrightarrow{M} E')$ is a linear map from $\mathcal{H}(E)$ to $\mathcal{H}(E')$.

If one prefers to work with cobordisms rather than weighted cobordisms (this is done in [I]–[IV]), one can start from a category $\mathsf{3Cob}'(\mathcal{C})$ which has the same objects as $\mathsf{3Cob}(\mathcal{C})$, but the morphisms are now only cobordisms, without the integer m. One can then define $(Z',\mathcal{H}') : \mathsf{3Cob}'(\mathcal{C}) \to \mathcal{V}ect_f(\Bbbk)$ in terms of $(Z,\mathcal{H})$ via $\mathcal{H}'(E) = \mathcal{H}(E)$ and $Z'(M) = Z((M,0))$. The pair $(Z',\mathcal{H}')$ is now only a projective functor in the sense that

$$Z'(M' \circ M) \;=\; \kappa^{\mu}\, Z'(M') \circ Z'(M) \quad , \tag{3.16}$$

where $M : E \to E'$, $M' : E' \to E''$ are cobordisms, $\kappa \in \Bbbk^\times$ and μ is an integer computed from the Lagrangian subspaces in E, E' and E'' via Maslov indices; for details see [Tu, section IV.7] or [FFFS, section 2.7]. The rank $\mathcal{D}$ of $\mathcal{C}$ one has to choose enters in the definition of κ, cf. [Tu, Theorem 7.1].

3.5 Combinatorial data of modular tensor categories

For explicit computations it is helpful to encode the tensor product and the braiding of a modular tensor category $\mathcal{C}$ in terms of fusing and braiding matrices, whose entries are numbers (i.e. elements of $\Bbbk$). In fact, this combinatorial data (or *chiral data*) had been extracted from

[5] In principle we should take the morphisms of $\mathsf{3Cob}$ to be equivalence classes of cobordisms, as we did for similar categories in chapter 2. In order to keep the notation simple, we will instead use the cobordisms directly, but the reader may also think of the morphisms of $\mathsf{3Cob}$ as equivalence classes.

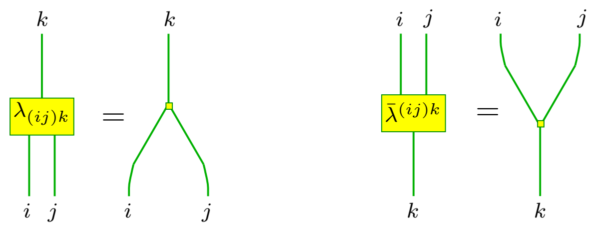
considerations in chiral CFT [MS1, MS3] before the notion of a modular tensor category had been introduced.

For each $i \in \mathcal{I}$, choose once and for all a representative U_i of the isomorphism class of simple objects labelled by i. Define

$$N_{ij}^{\;k} = \dim_{\Bbbk} \mathrm{Hom}(U_i \otimes U_j, U_k) \quad . \tag{3.17}$$

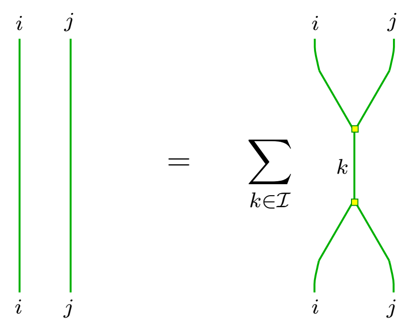
To avoid a large number of indices, in this section we make the simplifying assumption that $N_{ij}^{\;k} \in \{0,1\}$. The notation with all indices in place can be found in section I:2.2.

To define the fusing and braiding matrices, first choose a basis vector $\lambda_{(ij)k}$ in each space $\mathrm{Hom}(U_i \otimes U_j, U_k)$ for which $N_{ij}^{\;k} = 1$. This choice also determines an element of $\bar\lambda^{(ij)k} \in \mathrm{Hom}(U_k, U_i \otimes U_j)$ via the requirement $\bar\lambda^{(ij)k} \circ \lambda_{(ij)k} = id_{U_k}$. The morphism $\bar\lambda^{(ij)k}$ will be referred to as the dual basis to $\lambda_{(ij)k}$. In the special case $i = 0$ or $j = 0$ we require $\lambda_{(i0)i} = id_{U_i}$ and $\lambda_{(0j)j} = id_{U_j}$. For the morphisms $\lambda_{(ij)k}$ and $\bar\lambda^{(ij)k}$ we use the graphical notation



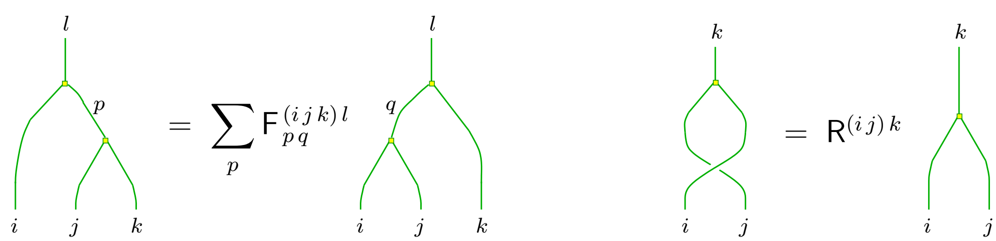
(3.18)

By semisimplicity we have the relation



(3.19)

The fusing matrices F and the braiding matrices R are defined via the relations



(3.20)

In the first equation, the numbers $\mathsf{F}_{p\,q}^{(i\,j\,k)\,l}$ are the coefficients expressing vectors of one basis of $\mathrm{Hom}(U_i \otimes U_j \otimes U_k, U_l)$ in terms of another. In the second equation, the number $\mathsf{R}^{(i\,j)\,k}$ is the multiplicative constant relating one nonzero element of $\mathrm{Hom}(U_i \otimes U_j, U_k)$ to another.

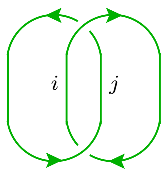
Define the numbers $\theta_i \in \Bbbk^\times$ via $\theta_{U_i} = \theta_i id_{U_i}$. Together with the index set $\mathcal{I}$ and the fusion multiplicities $N_{ij}^{\;k}$, the constants

$$\dim(U_i) \;, \quad \theta_i \;, \quad \mathsf{F}_{p\,q}^{(i\,j\,k)\,l} \;, \quad \mathsf{R}^{(i\,j)\,k} \tag{3.21}$$

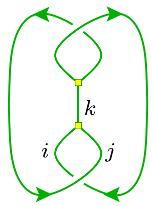
fully encode the modular tensor category. However they do so in a highly non-canonical way, as many basis choices had to be made.

The invariants associated to ribbon graphs via the 3dTFT can be expressed in terms of the combinatorial data. In particular, the s-matrix is obtained as

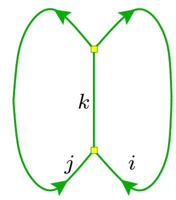
$$s_{i,j} =$$



$$= \sum_{k\in\mathcal{I}}$$

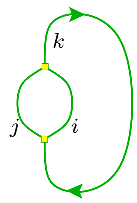


$$= \sum_{k\in\mathcal{I}} \mathsf{R}^{(i\,j)\,k}\mathsf{R}^{(j\,i)\,k}$$



(3.22)

$$= \sum_{k\in\mathcal{I}} \mathsf{R}^{(i\,j)\,k}\mathsf{R}^{(j\,i)\,k}$$



$$= \sum_{k\in\mathcal{I}} N_{ij}^{\;k}\,\frac{\theta_k}{\theta_i\,\theta_j}\,\dim(U_k) \quad ,$$

where we used (3.19), the definition (3.20) of the braiding matrices (as well as the corresponding dual relation, see appendix II:A.1 for a collection of rules), and equation (I:2.43) for the product of two braid matrices.

For example, for $\mathcal{C}(G,\psi,\Omega)$ (or rather, its strict version $\mathcal{C}^{\rm str}(G,\psi,\Omega)$) we have $\mathcal{I}=G$ and, for $a,b,c\in G$, $N_{ab}^{\;c}=\delta_{a\cdot b,c}$ as well as (for an appropriate choice of bases in the Hom-spaces),

$$\dim(U_a)=1\ ,\quad \theta_a=\Omega(a,a)^{-1}\ ,\quad \mathsf{F}^{(a\,b\,c)\,a\cdot b\cdot c}_{b\cdot c,\;a\cdot b}=\psi(a,b,c)^{-1}\ ,\quad \mathsf{R}^{(a\,b)\,a\cdot b}=\Omega(b,a)^{-1}\quad , \tag{3.23}$$

where '$\cdot$' denotes the product in G. More details can be found in remark III:2.12. As another example, the combinatorial data for the representation category of the affine Lie algebra $\widehat{su}(2)_k$ is summarised in section I:2.5.2.

3.6 Mapping class group and factorisation

An *isomorphism of extended surfaces* $f:E\to E'$ is an orientation preserving (degree one) homeomorphism from E to E' compatible with the marked points and Lagrangian subspaces. That is, if E has n marked points, there is a numbering of the marked points of E' s.t. $(p_i',[\gamma_i'],V_i',\varepsilon_i')=(f(p_i),[f\circ\gamma_i],V_i,\varepsilon_i)$; for the Lagrangian subspaces we require $\lambda_{E'}=f_*\lambda_E$.

Mapping class group

Given an isomorphism of extended surfaces $f:E\to E'$ we can construct a cobordism $M_f: E\to E'$ by setting

$$M_f \;=\; \big(E\times[-1,0]\big)\;\sqcup\;\big(E'\times[0,1]\big)/\sim \tag{3.24}$$

where the equivalence relation is given by $(e,0)\sim(f(e),0)$ for all $e\in E$. By construction, if f and f' are homotopic, then so are M_f and $M_{f'}$. Choosing the weight to be zero, we obtain a morphism $M_f\equiv(M_f,0)$ in $\mathsf{3Cob}(\mathcal{C})$. Applying the functor $(Z,\mathcal{H})$ results in a linear map $Z(M_f):\mathcal{H}(E)\to\mathcal{H}(E')$. We will also use the notation $f_\sharp:=Z(M_f)$.

In the special case $E = E'$ this gives rise to a projective representation ρ_E of the mapping class group $\mathrm{Map}(E)$ of E on $\mathcal{H}(E)$. If f is a representative of an element $[f] \in \mathrm{Map}(E)$ we set $\rho_E([f]) = Z(M_f)$. One then checks

$$\rho_E([f])\, \rho_E([g]) = Z(M_f)\, Z(M_g) = Z(M_f \circ M_g) = \kappa^\mu\, Z(M_{f \circ g}) \quad . \tag{3.25}$$

The factor κ^μ arises because the weighted cobordism $M_f \circ M_g$ will in general not have weight zero, while $M_{f \circ g}$ has weight zero by definition. For computations of the power μ see e.g. [Tu, section IV.5] or appendix II:A.3 where the torus is treated.

Since the mapping class group of a torus T without marked points is the modular group $\mathrm{PSL}(2,\mathbb{Z})$, a modular tensor category provides in particular a projective representation of the modular group. This is the reason for the name. Denote the standard generators of $\mathrm{PSL}(2,\mathbb{Z})$ by

$$\tilde{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad , \qquad \tilde{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad . \tag{3.26}$$

The space $\mathcal{H}(\mathrm{T})$ has dimension $|\mathcal{I}|$ and it comes with a natural basis $\{\, |\chi_i; \mathrm{T}\rangle \,|\, i \in \mathcal{I} \,\}$, see (I:5.15) for a figure and appendix II:A.3 for more details. In terms of this basis, the elements $\tilde{S}, \tilde{T}$ get represented by $\mathcal{I}{\times}\mathcal{I}$-matrices,

$$\big(\rho_{\mathrm{T}}(\tilde{S})\big)_{ij} \;=\; \hat{S}_{ij} := \mathcal{D}^{-1}\, s_{i,j} \qquad , \qquad \big(\rho_{\mathrm{T}}(\tilde{T})\big)_{ij} \;=\; \hat{T}_{ij} := \delta_{i,j}\, \theta_i^{-1} \quad , \tag{3.27}$$

where $s_{i,j}$ is the s-matrix (3.11) and the $\theta_i \in \Bbbk^\times$ are the defined via $\theta_{U_i} = \theta_i\, id_{U_i}$ as above.

Factorisation

Let $A_\varepsilon \subset \mathbb{R}^2$ be an annulus as defined in section 2.4. Denote by $D_{1+\varepsilon}(U)^\pm \subset \mathbb{R}^2$ a disc of radius $1{+}\varepsilon$ with a marked point $(0, [\gamma], U, \pm)$, where $\gamma(t) = t$.

Given an extended surface E and an embedding $f : A_\varepsilon \to E$ such that $\mathrm{Im}(f)$ contains no marked point of E, we can define a new extended surface $\Gamma_{f,U}(E)$ as follows. Let $E' = E \backslash f(S^1)$ be the surface E minus the image of the unit circle under f. Then

$$\Gamma_{f,U}(E) = E' \sqcup D^+_{1+\varepsilon}(U) \sqcup D^-_{1+\varepsilon}(U) / \sim \quad , \tag{3.28}$$

where, if we identify $\mathbb{R}^2 \equiv \mathbb{C}$, $z \sim f(z)$ for all $z \in A^+_\varepsilon \subset D^+_{1+\varepsilon}(U)$ and $z \sim f(-1/z)$ for all $z \in A^+_\varepsilon \subset D^-_{1+\varepsilon}(U)$. The Lagrangian subspace for $\Gamma_{f,U}(E)$ will be given below.

In words, this amounts to removing an S^1 from E and gluing two half discs into the resulting holes, one with a marked point $(U,+)$ and the other with a marked point $(U,-)$.

From such a cutting procedure one also obtains a linear map $g_{f,U}(E) : \mathcal{H}(\Gamma_{f,U}(E)) \to \mathcal{H}(E)$, a *gluing homomorphism*, as follows. Consider the cobordism

$$M_{f,U}(E) = \Gamma_{f,U}(E) \times [0,1] / \sim \tag{3.29}$$

where the equivalence relation identifies two subsets of $\Gamma_{f,U}(E) \times \{1\}$. Specifically, for all $z \in \mathbb{C}$ with $|z| \leq 1$, we identify $z \in D^+_{1+\varepsilon}(U) \subset \Gamma_{f,U}(E)$ with $-z^* \in D^-_{1+\varepsilon}(U) \subset \Gamma_{f,U}(E)$. This identifies the two discs just added, so that they are not part of the boundary of $M_{f,U}(E)$ and we obtain a cobordism $\Gamma_{f,U}(E) \to E$, see [FFRS] for details and figures.

The cobordism $M_{f,U}(E)$ also defines the Lagrangian subspace λ' of the extended surface $\Gamma_{f,U}(E)$ by taking λ' to consist of all elements $x' \in H_1(\Gamma_{f,U}(E),\mathbb{R})$ for which that there exists an element $x \in \lambda_E$ such that $x' - x$ is contractible in $M_{f,U}(E)$. It is shown in [Tu, section IV.4.2] that this indeed defines a Lagrangian subspace.

Choosing $M_{f,U}(E)$ to have weight zero, we obtain a morphism $\Gamma_{f,U}(E) \to E$ in $\mathsf{3Cob}(\mathcal{C})$. The gluing homomorphism is then just defined as $g_{f,U}(E) = Z(M_{f,U}(E))$. The factorisation relation between the spaces of states of the 3dTFT is that [Tu, Lemma IV.2.2.2]

$$\bigoplus_{i\in\mathcal{I}} g_{f,U_i}(E) \;:\; \bigoplus_{i\in\mathcal{I}} \mathcal{H}(\Gamma_{f,U_i}(E)) \longrightarrow \mathcal{H}(E) \tag{3.30}$$

is an isomorphism.

Remark 3.2:
The gluing homomorphisms are needed to formulate the factorisation condition on CFT correlators – already described in section 2.5 – in the algebraic construction of a consistent set of correlators to be formulated in chapter 6.

4 Relating 3dTFT and 2dCFT

4.1 Vertex algebras and conformal blocks

In section 2.5 the notion of holomorphic fields was introduced. It was already mentioned in the introduction that a conformal vertex algebra is a formalisation of the properties of holomorphic fields. Vertex algebras where introduced by [B] and further studied in [FLM]. As there are already several books on this subject [FB, Hu, Kc, LL], this section will be very brief. A slightly more extended summary is given in sections IV:5.1, IV:5.2 (the following is based on sections 1.2, 1.3, 2.5 and 5.1 of [FB]).

Vertex algebras and modules

Let $\mathrm{End}(V)[\![z]\!]$, for V a vector space, denote the space of formal power series in the indeterminate z with coefficients in $\mathrm{End}(V)$. Suppose $V = \bigoplus_{n\in\mathbb{Z}} V^{(n)}$ is $\mathbb{Z}$-graded. A *field of conformal dimension* $\Delta \in \mathbb{Z}$ is a formal power series $A(z) = \sum_{j\in\mathbb{Z}} A_j z^{-j} \in \mathrm{End}(V)[\![z, z^{-1}]\!]$ s.t. $A_j(V^{(n)}) \subset V^{(n-j+\Delta)}$ for all n and, for any $v \in V$, $A_j(v) = 0$ for large enough j.

A *conformal vertex algebra* $\mathfrak{V}$ *of central charge* c consists of the following data. There is a vector space R_Ω, the space of states, which is $\mathbb{Z}_{\geq 0}$-graded with finite-dimensional homogeneous components. We have a state-field correspondence Y, assigning to every $W \in \mathrm{R}_\Omega$ a field $Y(W;z) \in \mathrm{End}(\mathrm{R}_\Omega)[\![z^{\pm 1}]\!]$. There are two distinguished vectors in R_Ω, the vacuum state $v_\Omega \in \mathrm{R}_\Omega^{(0)}$ and the Virasoro vector $v_{\mathrm{vir}} \in \mathrm{R}_\Omega^{(2)}$.

This data is subject to the following conditions. The field associated to the vacuum state is the identity, $Y(v_\Omega;z) = id_{\mathrm{R}_\Omega}$, and $Y(u;z)v_\Omega = u + O(z)$ for all $u \in \mathrm{R}_\Omega$. The coefficients L_m in $T(z) := Y(v_{\mathrm{vir}};z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}$ obey the Virasoro algebra of central charge c,

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{c}{12}\delta_{m+n,0}(m^3-m) \quad . \tag{4.1}$$

Further, L_0 gives the grading on R_Ω, i.e. $L_0\mathrm{R}_\Omega^{(n)} = n\mathrm{R}_\Omega^{(n)}$ and L_{-1} is the generator of translations, $[L_{-1}, Y(W,z)] = \partial_z Y(W,z)$ for all $W \in \mathrm{R}_\Omega$. The field $T(z)$ is called the (holomorphic component of the) *stress tensor*. Finally, and most important, all fields $Y(\cdot;z)$ are *local*, i.e. for all $A, B \in \mathrm{R}_\Omega$ one may find an $N \in \mathbb{Z}_{\geq 0}$ s.t.

$$(z-w)^N \big[Y(A;z), Y(B;w) \big] = 0 \tag{4.2}$$

as a formal power series in $\mathrm{End}(\mathrm{R}_\Omega)[\![z^{\pm 1}, w^{\pm 1}]\!]$.

A (graded) *module* of a conformal vertex algebra $\mathfrak{V}$ is a $\mathbb{C}$-graded vector space R equipped with a map $Y_{\mathrm{R}}(\,\cdot\,;z) : \mathrm{R}_\Omega \to \mathrm{End}(\mathrm{R})[\![z^{\pm 1}]\!]$, which assigns to every $A \in \mathrm{R}_\Omega^{(m)}$ a field $Y_{\mathrm{R}}(A;z)$ of conformal dimension m, and which is compatible with the structure of the conformal vertex algebra $\mathfrak{V}$, see e.g. section 5.1 of [FB] for details. In particular, $\mathfrak{V}$ will be a module over itself. If this module is simple, then we call also $\mathfrak{V}$ *simple*.

Given a $\mathfrak{V}$-module R, we define its *character* $\chi_{\mathrm{R}}(q)$ to be the formal power series given by the trace

$$\chi_{\mathrm{R}}(q) = \mathrm{tr}_{\mathrm{R}}\, q^{L_0 - c/24} \quad . \tag{4.3}$$

If we substitute $q \in \mathbb{C}$, as we will do below, the result may be infinite, but for the examples coming from CFT this series converges for $|q| < 1$.

An important class of examples is provided by conformal vertex algebras $\mathfrak{V}(\widehat{g}_k)$ constructed from affine Lie algebras $\widehat{g}_k$; in this case, R_Ω is given by the integrable highest weight representation with highest weight zero. The simple modules are given by irreducible integrable highest weight representations of $\widehat{g}_k$ and the characters of these representations are provided by the Weyl-Kac character formula, see e.g. [FB, example 5.5.5] for more details and references.

Extended Riemann surfaces and conformal blocks

An *extended Riemann surface* E^c is a compact Riemann surface with empty boundary, also denoted by E^c, together with a finite ordered set of marked points $(p_i, [\varphi_i], \mathrm{R}_i)$, where $p_i \in E^c$ are mutually distinct points, $[\varphi_i]$ is a germ of injective holomorphic functions from a small disk $D_\delta \subset \mathbb{C}$ around 0 to E^c such that $\varphi_i(0) = p_i$, and R_i is a module of the conformal vertex algebra $\mathfrak{V}$. Finally, E^c is equipped with a Lagrangian submodule $\lambda^c \subset H_1(E^c, \mathbb{Z})$ (see section IV:5.2).

Let E^c be an extended Riemann surface with n marked points. The *space of conformal blocks* $\mathcal{H}^c(E^c)$ is defined as a subspace of $(\mathrm{R}_1 \otimes \cdots \otimes \mathrm{R}_n)^*$, i.e. of the space of multi-linear functions $\mathrm{R}_1 \times \cdots \times \mathrm{R}_n \to \mathbb{C}$. The subspace is characterised by a somewhat involved compatibility condition with the action of $\mathfrak{V}$ (the difficulty is to remove the local coordinate in the definition of the action of $\mathfrak{V}$), details can be found in e.g. in section 10.1 of [FB].

Modular tensor categories

In order to make the connection to 3dTFT later on, we need to restrict our attention to a subclass of conformal vertex algebras, namely to those, whose category $\mathcal{R}ep(\mathfrak{V})$ of $\mathfrak{V}$-modules is a modular tensor category.

A sufficient set of conditions on a vertex algebra $\mathfrak{V}$ for $\mathcal{R}ep(\mathfrak{V})$ to be modular have been given in [Hu1, Theorem 3.1], (building on earlier work [HL] defining a braided tensor structure on $\mathcal{R}ep(\mathfrak{V})$): $\mathfrak{V}$ should be simple, have $\mathrm{R}_\Omega^{(0)} = \mathbb{C}v_\Omega$, and obey Zhu's C_2-cofiniteness condition.

Further, every graded $\mathfrak{V}$-module R has to be completely reducible, and $\mathrm{R}^{(0)} = 0$ for any irreducible $\mathfrak{V}$-module not isomorphic to $\mathfrak{V}$.

Given a conformal vertex algebra $\mathfrak{V}$ s.t. $\mathcal{R}ep(\mathfrak{V})$ is modular, we choose a set $\{\, \mathrm{S}_i \,|\, i \in \mathcal{I} \,\}$ of representatives for the isomorphism classes of simple $\mathfrak{V}$-modules. In particular, the index set $\mathcal{I}$ is finite.

It can also be shown that under the above assumptions, the characters $\chi_i \equiv \chi_{\mathrm{S}_i}(q)$ form a $|\mathcal{I}|$-dimensional representation of the modular group [Z,DLM]. In particular, setting $q = \exp(2\pi i\tau)$, there are matrices S_{ij} and T_{ij} such that

$$\chi_i\big(e^{2\pi i(\tau+1)}\big) = \sum_{j\in\mathcal{I}} T_{ij}\ \chi_j(q) \qquad \text{and} \qquad \chi_i\big(e^{2\pi i(-1/\tau)}\big) = \sum_{j\in\mathcal{I}} S_{ij}\ \chi_j(q) \ . \tag{4.4}$$

Remark 4.1:
Recall that in (3.27) we have already met a $|\mathcal{I}|$-dimensional representation of the modular group defined in terms of a modular tensor category. As will be stressed in section 4.3 below, the structure of a modular tensor category on $\mathcal{R}ep(\mathfrak{V})$ is defined entirely in terms of genus zero conformal blocks. In particular, as stated in (3.27), the quantities $\hat{S}_{ij}$ and $\hat{T}_{ij}$ are given by the invariant of the Hopf link (3.11) and by the twist of the modular tensor category, respectively, and not in terms of transformation properties of characters. It is a quite non-trivial fact, known as *Verlinde conjecture* [Ve] and proved under the above assumptions in [Hu1] (the proof being a rigorous version in the framework of vertex algebras of the CFT-based argument in [MS1,MS3]) that the matrices in (3.27) and (4.4) are simply related by

$$S \ = \ \hat{S} \qquad \text{and} \qquad T \ = \ e^{-2\pi i c/24}\ \hat{T}_{ij} \ , \tag{4.5}$$

with the phase accounting for the fact that the representation generated by $\hat{S}$ and $\hat{T}$ is only projective. The natural setting in which to understand this surprising relation is the (also conjectured) deep relation between the space of conformal blocks of $\mathfrak{V}$ on a given extended Riemann surface and the corresponding space of states of the TFT associated to $\mathcal{R}ep(\mathfrak{V})$, see section 4.3 below.

4.2 Correlators and conformal blocks on the double

Holomorphic fields on the Riemann sphere

Given a conformal vertex algebra $\mathfrak{V}$, we would like to construct a CFT which contains $\mathfrak{V}$ as a subspace of its holomorphic fields. More precisely, recall from section 2.5 that the space of fields $\mathcal{F} \subset \mathsf{H}$ was defined as the direct sum of graded components of H. Let the world sheet X^g be a sphere with some metric g, an orientation or_2 and n marked points. Let E^c be the extended Riemann surface which is given by X^g, considered as a complex manifold (i.e. the Riemann sphere), with each marked point in addition labelled by the vacuum module R_Ω of $\mathfrak{V}$. In this case the space of conformal blocks $\mathcal{H}^c(E^c)$ is one-dimensional (see e.g. [FB, section 10.4]).

We say that a CFT *has* $\mathfrak{V}$ *as chiral algebra* if there is an embedding $\iota : \mathrm{R}_\Omega \to \mathcal{F}$ such that the image of ι lies in the subspace of holomorphic fields, and such that for X^g as described above, and for $v_1,\dots,v_n \in \mathrm{R}_\Omega$ we have

$$\frac{C(\mathrm{X}^g)(u_1,\dots,u_n)}{C(X^g)(u_0,\dots,u_0)} = \frac{\beta(v_1,\dots,v_n)}{\beta(v_\Omega,\dots,v_\Omega)} \quad , \tag{4.6}$$

where $u_k = \iota(v_k)$, $u_0 = \iota(v_\Omega)$, and β is any nonzero element of $\mathcal{H}^c(E^c)$. That is, the normalised correlators of holomorphic fields (in the image of ι) on the Riemann sphere are just given by the unique (normalised) conformal block that is available.

In the same way we say that a CFT has $\mathfrak{V}$ as anti-chiral algebra if there is an embedding $\bar\iota : \mathrm{R}_\Omega \to \mathcal{F}$ that lies entirely in the subspace of anti-holomorphic fields, and such that (4.6) holds for a nonzero block $\beta \in \mathcal{H}^c(\bar E^c)$, where $\bar E^c$ is obtained from X^g with orientation $-\mathrm{or}_2$ instead of or_2.

We will only consider CFTs which have a given conformal vertex algebra $\mathfrak{V}$ both as chiral and anti-chiral algebra. The notion of a module over $\mathfrak{V}$ is tailored to imply that the space of bulk fields $\mathcal{F}$ is in fact a module of $\mathfrak{V}\times\mathfrak{V}$ (the action of the chiral and anti-chiral algebra). Further, we demand that $\mathcal{R}ep(\mathfrak{V})$ is modular. It is then reasonable to make the ansatz

$$\mathcal{F} = \bigoplus_{i,j\in\mathcal{I}} \big(\mathrm{S}_i \otimes \mathrm{S}_j\big)^{\oplus Z_{ij}} \quad , \tag{4.7}$$

for some non-negative integers Z_{ij}, and where S_i refer to the simple $\mathfrak{V}$-modules.

For the space boundary fields $\mathcal{F}_{ab}$ the situation is different. Recall from section 2.6 that for a conformal boundary condition, given $q\in\partial X$, we have $T(q) = \bar T(q)$ inside any correlator. In order to make full use of the chiral algebra $\mathfrak{V}$ we have to restrict ourselves to boundary conditions that *preserve* $\mathfrak{V}$, i.e. for all $v\in\mathfrak{V}$ we require $W(q) = \bar W(q)$ where $W = \iota(v)$, $\bar W = \bar\iota(v)$ and $q\in\partial\mathrm{X}^g$.

From this argument one sees that the two copies $\mathfrak{V}\times\mathfrak{V} \hookrightarrow \mathcal{F}$ actually act in the same way on boundary fields (an observation first made in [C1]) and hence

$$\mathcal{F}_{ab} = \bigoplus_{i\in\mathcal{I}} \mathrm{S}_i^{\oplus A_{ia}^{\ \ b}} \tag{4.8}$$

for some non-negative integers $A_{ia}^{\ \ b}$.

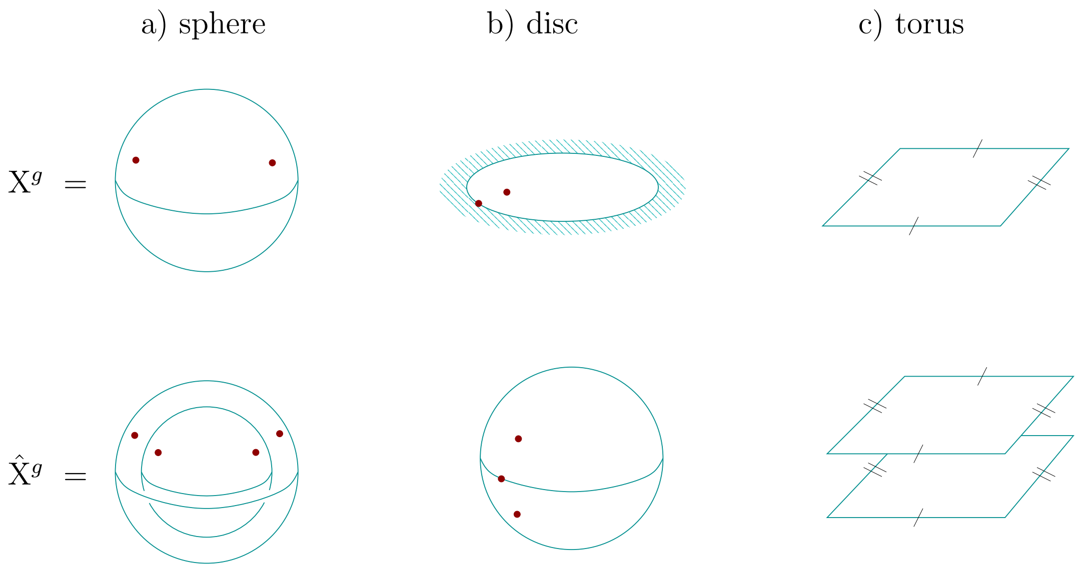
A CFT with chiral algebra $\mathfrak{V}(\widehat{g}_k)$ is called a g-WZW model at level k, see e.g. [DMS, chapter 15]. Next to the Virasoro minimal models of [BPZ], WZW-models are the best studied conformal field theories.

Holomorphic factorisation

Let X^g be a world sheet, possibly with boundaries, possibly unoriented. One can construct an extended Riemann surface $\hat{\mathrm{X}}^g$, the *complex double of* X^g by taking the orientation bundle over X^g, divided by an equivalence relation,

$$\hat{\mathrm{X}}^g = \mathrm{Or}(\mathrm{X}^g)/\!\sim \qquad \text{with} \;\; (x,\mathrm{or}_2) \sim (x,-\mathrm{or}_2) \quad \text{for} \;\; x\in\partial\mathrm{X}^g\,. \tag{4.9}$$

Since a metric (or just a conformal structure) together with an orientation defines a complex structure, we obtain a complex structure on $\hat{\mathrm{X}}^g$. Note also that by construction, $\hat{\mathrm{X}}^g$ has an empty boundary. Further, there is a projection $\pi : \hat{\mathrm{X}}^g \to \mathrm{X}^g$ taking $[x, \mathrm{or}_2]$ to x. The marked points on $\hat{\mathrm{X}}^g$ are obtained by taking the pre-images of the marked points on X^g under π, see section IV:6.1 for more details, where also the Lagrangian submodule for $\hat{\mathrm{X}}^g$ is defined. In particular, a bulk insertion on X^g leads to two marked points on $\hat{\mathrm{X}}^g$, and a boundary insertion on X^g leads to one marked points on $\hat{\mathrm{X}}^g$. Some examples of X^g and the resulting $\hat{X}^g$ are



(4.10)

Examples where X^g is non-orientable can be found in sections II:3.3 and II:3.5.

The significance of conformal blocks in the construction of correlators of a CFT with chiral (and anti-chiral) algebra $\mathfrak{V}$ is the fact that a correlator on X^g is an element in the space of conformal blocks for $\mathfrak{V}$ on the double $\hat{\mathrm{X}}^g$,

$$C(\mathrm{X}^g) \in \mathcal{H}^c(\hat{\mathrm{X}}^g) \quad . \tag{4.11}$$

This is the principle of *holomorphic factorisation* [Wt3]. Note that we have already met one instance of this condition in (4.6). Let us consider some more examples of this important principle.

Oriented surfaces without boundaries

Consider the situation in figure (4.10 a), where X^g is a sphere with, say, n bulk field insertions. Recall from section 2.5 that in this case $C(X)$ is a functional $\mathcal{F}^{\otimes n} \to \mathbb{C}$. Suppose that at the k'th marked point we only insert fields in the component $\mathrm{S}_{i_k} \otimes \mathrm{S}_{j_k}$ of $\mathcal{F}$, i.e. we restrict the functional $C(X)$ to

$$C(X) : \big(\mathrm{S}_{i_1} {\otimes} \mathrm{S}_{j_1}\big) \otimes \cdots \otimes \big(\mathrm{S}_{i_n} {\otimes} \mathrm{S}_{j_n}\big) \longrightarrow \mathbb{C} \quad . \tag{4.12}$$

The double of an oriented surface $(\mathrm{X}^g, \mathrm{or})$ with empty boundary consists of two copies of that surface with opposite orientations $\hat{\mathrm{X}}^g = (\mathrm{X}^g, \mathrm{or}) \sqcup (\mathrm{X}^g, -\mathrm{or})$. Analogous to the discussion in the beginning of the section, denote by E^c and $\bar{E}^c$ the extended Riemann surfaces obtained by taking $(\mathrm{X}^g, \mathrm{or})$ and $(\mathrm{X}^g, -\mathrm{or})$ (both of which are isomorphic to the Riemann sphere as complex curves), and labelling the marked points by $\mathrm{S}_{i_1}, \dots, \mathrm{S}_{i_n}$ and $\mathrm{S}_{j_1}, \dots, \mathrm{S}_{j_n}$, respectively. The space of conformal blocks on $\hat{\mathrm{X}}^g$ is then $\mathcal{H}^c(\hat{\mathrm{X}}^g) = \mathcal{H}^c(E^c) \otimes \mathcal{H}^c(\bar{E}^c)$ and holomorphic factorisation states that there exist $\beta_\alpha \in \mathcal{H}^c(E^c)$ and $\beta'_\alpha \in \mathcal{H}^c(\bar{E}^c)$ s.t.

$$C(X)(v_{i_1} \otimes v_{j_1}, \dots, v_{i_n} \otimes v_{j_n}) = \sum_\alpha \beta_\alpha(v_{i_1}, \dots, v_{i_n}) \beta'_\alpha(v_{j_1}, \dots, v_{j_n}) \tag{4.13}$$

for any $v_{i_k} \in \mathrm{S}_{i_k}$ and $v_{j_k} \in \mathrm{S}_{j_k}$.

Surfaces with boundaries

Let now X^g be a disc as in figure (4.10 b), with m boundary field insertions and n bulk field insertions. In this case the correlator $C(X)$ is a functional of the form

$$C(X) : \mathcal{F}_{a_1 b_1} \otimes \mathcal{F}_{a_2 b_2} \cdots \otimes \mathcal{F} \otimes \mathcal{F} \cdots \longrightarrow \mathbb{C} \quad . \tag{4.14}$$

Restricting to the components $\mathrm{S}_{r_k} \subset \mathcal{F}_{a_k b_k}$ of the k'th boundary field and to $\mathrm{S}_{i_l} \otimes \mathrm{S}_{j_l} \subset \mathcal{F}$ of the l'th bulk field we obtain a functional

$$C(\mathrm{X}^g) : \quad \bigotimes_{k=1}^{m} \mathrm{S}_{r_k} \otimes \bigotimes_{l=1}^{n} (\mathrm{S}_{i_l} \otimes \mathrm{S}_{j_l}) \longrightarrow \mathbb{C} \,. \tag{4.15}$$

According to the construction of the double in (4.10) this now has to be an element in the space of conformal blocks on the Riemann sphere with $m + 2n$ marked points (again, this was first observed in [C1]).

Torus

Finally, for the third case in figure (4.10), let T_τ be the torus obtained by dividing $\mathbb{C}$ by the lattice spanned by the vectors 1 and τ, where τ lies in the upper half plane. On T_τ we take the metric and complex structure induced by $\mathbb{C}$. The space of conformal blocks on the torus is spanned by the characters of the irreducible representations of $\mathfrak{V}$,

$$\mathcal{H}^c(T_\tau) = \mathrm{span}_{\mathbb{C}} \big\{ \chi_{\mathrm{S}_i}(q) \big| i \in \mathcal{I} \big\} \quad , \tag{4.16}$$

where $q = \exp(2\pi i \tau)$. Note that two tori T_τ and $T_{\tau'}$ describe the same elliptic curve iff τ and τ' are related by a modular transformation $\tau' = (a\tau + b)/(c\tau + d)$ (i.e. $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$). That modular transformations leave the space $\mathcal{H}^c(T_\tau)$ invariant is guaranteed by the modular transformation properties (4.4) of the characters.

Holomorphic factorisation requires $C(T_\tau) \in \mathcal{H}^c(T_\tau) \otimes \mathcal{H}^c(T_{-\tau^*})$. In fact, using the property that in the functorial description of a CFT, the trace in $\mathsf{2Rie}$ gets mapped to the trace of vector spaces, one finds that $C(T_\tau)$ is determined by the ansatz (4.7) to be

$$C(T_\tau) = \sum_{i,j \in \mathcal{I}} Z_{ij} \; \chi_{\mathrm{S}_i}(q) \, \chi_{\mathrm{S}_j}(q^*) \tag{4.17}$$

where again $q = \exp(2\pi i \tau)$. The correlators $C(T_\tau)$ have to obey

$$C(T_\tau) = C(T_{\tau'}) \qquad \text{whenever} \quad \tau' = \frac{a\tau + b}{c\tau + d} \;\; \text{for} \;\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \quad , \tag{4.18}$$

To see this, first note that the pairs $(\tau, 1)$ and $(a\tau+b, c\tau+d)$ generate the same lattice in $\mathbb{C}$. The latter set of generators is related to $(\tau', 1)$ by a simple rescaling $\zeta \mapsto f(\zeta) = \frac{a\tau+b}{c\tau+d} \cdot \zeta$. In particular, f is a conformal transformation from T_τ to $T_{\tau'}$ and property (C3) in section 2.4 implies $C(T_\tau) = e^{cS_{\text{liou}}} C(T_{\tau'})$. Due to the simple form of f, one finds $S_{\text{liou}} \equiv 0$.

Since all modular transformations are generated by $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$, comparing to (4.4) we find that (4.18) holds if

$$[S, Z] = 0 = [T, Z] \quad . \tag{4.19}$$

If all characters $\chi_{S_i}(q)$ are linearly independent, we also have "only if" in the above statement. In this case, (4.19) is thus a necessary condition in order to have a consistent CFT. (In fact, considering also correlators with one bulk field insertion on the torus, one can show that (4.19) is always necessary).

It was first noted in [C2] that this can put strong constraints on the possible values for Z_{ij}. For the case of $su(2)$-WZW-models at level k, there is a classification theorem [CIZ] which states

Theorem 4.2:
All $k \times k$ matrices Z with non-negative integer entries, and which obey

$$Z_{00} = 1 \quad , \quad [S, Z] = 0 = [T, Z] \tag{4.20}$$

are given by the following list.

- *(A-series)* for all k:
$$Z_{ij} \;=\; \delta_{ij} \quad . \tag{4.21}$$
- *(D_{even}-series)* for $k \cong 0 \bmod 4$:
$$Z_{ij} \;=\; \delta_{i \in 2\mathbb{Z}} \left(\delta_{i,j} + \delta_{i,k-j} \right) \quad . \tag{4.22}$$
- *(D_{odd}-series)* for $k \cong 2 \bmod 4$:
$$Z_{ij} \;=\; \delta_{i \in 2\mathbb{Z}}\, \delta_{i,j} + \delta_{i \in 2\mathbb{Z}+1}\, \delta_{i,k-j} \quad . \tag{4.23}$$
- *(E_6)* for k=10:
$$Z_{ij} \;=\; (\delta_{i,0}+\delta_{i,6})(\delta_{j,0}+\delta_{j,6}) + (\delta_{i,3}+\delta_{i,7})(\delta_{j,3}+\delta_{j,7}) + (\delta_{i,4}+\delta_{i,10})(\delta_{j,4}+\delta_{j,10}) \tag{4.24}$$
- *(E_7)* for k=16:
$$\begin{aligned} Z_{ij} = {} & (\delta_{i,0}+\delta_{i,16})(\delta_{j,0}+\delta_{j,16}) \;+\; (\delta_{i,4}+\delta_{i,12})(\delta_{j,4}+\delta_{j,12}) \;+\; (\delta_{i,6}+\delta_{i,10})(\delta_{j,6}+\delta_{j,10}) \\ & + \delta_{i,8}\delta_{j,8} \;+\; \delta_{i,8}(\delta_{j,2}+\delta_{j,14}) \;+\; (\delta_{i,2}+\delta_{i,14})\delta_{j,8} \end{aligned} \tag{4.25}$$
- *(E_8)* for k=28:
$$\begin{aligned} Z_{ij} = {} & (\delta_{i,0}+\delta_{i,10}+\delta_{i,18}+\delta_{i,28})(\delta_{j,0}+\delta_{j,10}+\delta_{j,18}+\delta_{j,28}) \\ & +(\delta_{i,6}+\delta_{i,12}+\delta_{i,16}+\delta_{i,22})(\delta_{j,6}+\delta_{j,12}+\delta_{j,16}+\delta_{j,22}) \quad . \end{aligned} \tag{4.26}$$

It should be stressed, however, that (4.19) is only *necessary* and by no means sufficient to have a consistent CFT. For the $su(2)$–WZW models it turns out (combine Theorem 5.17 and Remarks 5.16 (ii) and 6.12 (iv) below) that all cases in Theorem 4.2 are realised, i.e. the modular invariant bilinear combinations of characters (4.17), with Z taken from Theorem 4.2, do indeed occur as the correlator on the torus of a full CFT. But examples of modular invariant combinations of characters are known (see e.g. [FSS]) which cannot occur as torus correlator of a full CFT.

4.3 Conformal blocks and 3dTFT

Now, given a $\mathfrak{V}$ s.t. $\mathcal{C} = \mathcal{R}ep(\mathfrak{V})$ is modular, we can apply the construction of section 3.4 to obtain a 3dTFT $(Z, \mathcal{H}) : \mathsf{3Cob}(\mathcal{C}) \to \mathcal{V}ect_f(\mathbb{C})$. In view of how the relation between chiral CFT and 3dTFT first arose in the case of Chern-Simons theory [Wt2, FK], one would expect that there are isomorphisms between the state spaces $\mathcal{H}(E)$ assigned to an extended surface by the 3dTFT and the spaces $\mathcal{H}^c(E^c)$ of conformal blocks for an extended Riemann surface E^c, which are compatible with the action of the mapping class group and with factorisation. However, in general the existence of such isomorphisms is not at all obvious, for the following reason.

Denote by $\mathbb{P}^1[\mathrm{R}_1, \dots, \mathrm{R}_n]$ the extended Riemann surface given by $\mathbb{P}^1$ with n marked points labelled by the $\mathfrak{V}$-modules $\mathrm{R}_1, \dots, \mathrm{R}_n$. The tensor product on $\mathcal{C} \equiv \mathcal{R}ep(\mathfrak{V})$ is defined in terms of conformal blocks on $\mathbb{P}^1$ [HL], for example one has

$$\dim \mathrm{Hom}_{\mathcal{C}}(\mathrm{R}_1 \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} \mathrm{R}_n, \mathrm{R}_\Omega) = \dim \mathcal{H}^c\big(\mathbb{P}^1[\mathrm{R}_1, \dots, \mathrm{R}_n]\big) \quad . \tag{4.27}$$

The braiding on $\mathcal{C}$ is defined in terms of analytic continuation of conformal blocks on $\mathbb{P}^1$ along a path which exchanges two of the marked points (i.e. each point of the path is an extended Riemann surface given by $\mathbb{P}^1$ with varying coordinates for two of the marked points).

Thus, by definition, the structure of a braided tensor category (and thus also that of a modular tensor category) on $\mathcal{C} = \mathcal{R}ep(\mathfrak{V})$ is entirely fixed by the genus zero conformal blocks. Now, as described in section 3.6, the 3dTFT provides a projective representation of the mapping class group on extended surfaces of every genus, and it also describes how the space of states behave under factorisation.

Analogously, projective representations of the mapping class group and the behaviour under factorisation can also be obtained directly from the spaces of conformal blocks on extended Riemann surfaces of higher genus.

The (open) question is now, what properties does one have to demand of $\mathfrak{V}$ to ensure that there is an identification of the spaces of states of the TFT obtained from $\mathcal{C} = \mathcal{R}ep(\mathfrak{V})$ with the spaces of conformal blocks of $\mathfrak{V}$, which is compatible with the projective action of the mapping class group and with the behaviour under factorisation, both of which are defined independently on the complex-analytic and the topological side.

Remark 4.3:
For surfaces of genus zero, such an identification exists basically by construction (compatibility with the action of the mapping class group is illustrated in sections IV:5.3 and IV:5.4). At genus one there is still some control due to the results mentioned in Remark 4.1. For genus ≥ 2, no criteria are known which guarantee the existence of such an identification.

As already stated in the introduction, the approach of the works [I]–[IV] is "top-down", in the sense that the aim is not to determine a precise set of conditions to gain control over the complex-analytic side of the construction of a CFT (desirable as it may be), but rather to separate the two problems by *assuming* that we are given a conformal vertex algebra $\mathfrak{V}$ with sufficiently nice properties. These properties are that

- $\mathcal{R}ep(\mathfrak{V})$ is a modular tensor category and for every extended Riemann surface E, the space of conformal blocks $\mathcal{H}^c(E)$ is finite-dimensional.
- for the 2dTFT $(Z, \mathcal{H})$ constructed from $\mathcal{R}ep(\mathfrak{V})$, there exists an identification between $\mathcal{H}(E)$

and $\mathcal{H}^c(E)$, which is compatible with the action of the mapping class group and with factorisation.
Let us refer to a conformal vertex algebra with these properties as a *rational chiral algebra*.

Borderline

We have now arrived at the borderline between the complex-analytic and the algebraic part of the construction of a CFT. The content of chapters 2 and 4 was mainly to motivate, and to state in a more or less precise way (limited by space and by the present day understanding of some of the issues involved) the properties one would like a CFT to have, as well as to describe the complex-analytic approach via vertex algebras.

The content of chapters 5 and 6, on the other hand, is entirely algebraic. A specific problem (Problem 6.6) in the setting of modular tensor categories (and their associated 3dTFTs) is solved (Theorem 6.11). From this point of view, the only purpose of chapters 2 and 4 is to show that this is indeed an interesting problem, via its relation to the construction of rational conformal field theories.

5 Algebra in braided tensor categories

Let us now turn to the study of algebras in braided tensor categories. We will focus on concepts needed to present the main results of [C]. Recall from chapter 3 that all categories are taken to be small.

5.1 Frobenius algebras

The notion of an algebra over a field $\Bbbk$ can be extended to general tensor categories in a straight-forward way [Pa1]. Let $\mathcal{C}$ be a tensor category with associator $\alpha_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ and unit constraints $\lambda_U : \mathbf{1} \otimes U \to U$, $\mu_U : U \otimes \mathbf{1} \to U$.

Definition 5.1:
An (associative) *algebra* (with unit) A in $\mathcal{C}$ is a triple (A, m, η) consisting of an object A of $\mathcal{C}$, a multiplication morphism $m \in \mathrm{Hom}(A \otimes A, A)$ and a unit morphism $\eta \in \mathrm{Hom}(\mathbf{1}, A)$, satisfying

$$m \circ (m \otimes id_A) = m \circ (id_A \otimes m) \circ \alpha_{A,A,A} \ , \quad m \circ (\eta \otimes id_A) = \lambda_A \ , \quad m \circ (id_A \otimes \eta) = \mu_A \, . \qquad (5.1)$$

Remark 5.2:
(i) We will often take $\mathcal{C}$ to be strict. In this case $\alpha_{A,A,A} = id_{A \otimes A \otimes A}$ and $\lambda_A = \mu_A = id_A$.
(ii) If we take $\mathcal{C}$ to be $\mathcal{V}ect_f(\Bbbk)$, then 5.1 is the usual definition of an unital associative $\Bbbk$-algebra.

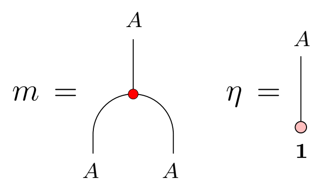
Example 5.3:
Let G be a finite abelian group. Consider the $\Bbbk$-vector space $A = \Bbbk[G]$. Define a bilinear multiplication $m : A \otimes A \to A$ via its values on a basis, $m(g, h) = \omega(g, h)\, g \cdot h$ with ω :

$G \times G \to \Bbbk^\times$ s.t. $\omega(g,e) = 1 = \omega(e,g)$ for all $g \in G$. Define also a unit map $\eta : \Bbbk \to A$ via $\eta(c) = c\,e$, with e the unit of G.
For (A, m, η) to be an associative unital algebra in $\mathcal{V}ect_f(\Bbbk)$ we need

$$\omega(f,g)\omega(fg,h) = \omega(g,h)\omega(f,gh) \tag{5.2}$$

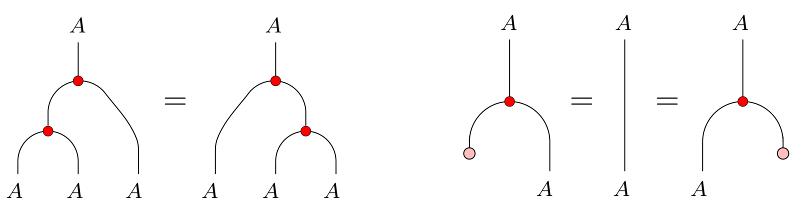
for all $f, g, h \in G$. Equivalently, in the terminology of group cohomology (see appendix III:A for conventions), ω has to be a normalised 2-cocycle on G with values in $\Bbbk^\times$. In this case A is a twisted group algebra.
On the other hand, if we define the 3-cocycle $\psi(f,g,h) = \mathrm{d}\omega(f,g,h)$, then for any normalised 2-cochain ω, (A, m, η) is an associative unital algebra in the category $\mathcal{C}(G, \psi, \Omega)$ as defined in section 3.3. This is true for any allowed choice of Ω; the braiding does not enter into the definition of an algebra. If however $\Omega(g,h) = \omega(g,h)/\omega(h,g)$ then (ψ, Ω) is an exact abelian 3-cocycle and the category $\mathcal{C}(G, \psi, \Omega)$ is equivalent to $\mathcal{C}(G, 1, 1)$ as a symmetric tensor category. Under this equivalence, the algebra A gets mapped to the (untwisted) group algebra $\Bbbk[G]$.

Let us now assume that $\mathcal{C}$ is a strict tensor category. It is helpful to introduce a pictorial notation for the morphisms m, η entering the definition of an algebra,



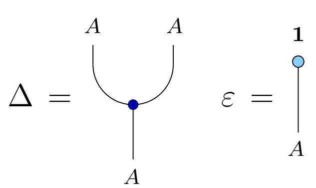
(5.3)

The associativity and unit conditions for m, η can then be depicted as



(5.4)

Analogously to an algebra (A, m, η) in $\mathcal{C}$ we define a *coalgebra* to be a triple (A, Δ, ε) obeying coassociativity and counit conditions, which amount to "turning (5.4) upside down". In particular, for Δ and ε we use the pictorial notation



(5.5)

The classical notions of Frobenius algebra, symmetric and special (cf. section 2.3) also extend to more general tensor categories in a straight-forward way.

Definition 5.4:
Let $\mathcal{C}$ be a strict tensor category.

(i) A *Frobenius algebra* in $\mathcal{C}$ is a quintuple $(A, m, \eta, \Delta, \varepsilon)$ such that (A, m, η) is an algebra in $\mathcal{C}$, (A, Δ, ε) is a co-algebra in $\mathcal{C}$, and there is the compatibility relation

$$(id_A \otimes m) \circ (\Delta \otimes id_A) = \Delta \circ m = (m \otimes id_A) \circ (id_A \otimes \Delta) \tag{5.6}$$

between the two structures.

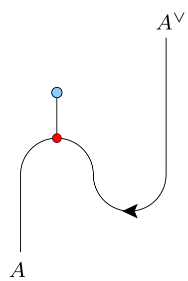
(ii) A Frobenius algebra is called *special* iff

$$\varepsilon \circ \eta = \beta_{\mathbf{1}}\, id_{\mathbf{1}} \qquad \text{and} \qquad m \circ \Delta = \beta_A\, id_A \tag{5.7}$$

for some $\beta_{\mathbf{1}}, \beta_A \in \mathbb{k}^{\times}$.

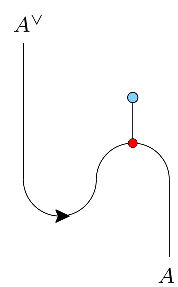
(iii) If $\mathcal{C}$ has left and right dualities, a Frobenius algebra in $\mathcal{C}$ is called *symmetric* iff the two morphisms

$$\Phi_1 := [(\varepsilon \circ m) \otimes id_{A^\vee}] \circ (id_A \otimes b_A) = \tag{5.8}$$



and

$$\Phi_2 := [id_{A^\vee} \otimes (\varepsilon \circ m)] \circ (\tilde{b}_A \otimes id_A) = \tag{5.9}$$



in $\mathrm{Hom}(A, A^\vee)$ are equal.

Recall that in the classical case, a Frobenius algebra A over $\mathbb{k}$ was defined to have a trace $\varepsilon : A \to \mathbb{k}$ which gives rise to a nondegenerate bilinear, invariant form on A. This characterisation is related to the above definition by Theorem 2.3. An analogous theorem (Theorem 5.5) exists also for tensor categories with dualities. It is proved in Lemma I:3.7.

Theorem 5.5:
Let $\mathcal{C}$ be a strict tensor category with left and right dualities. An algebra $A \equiv (A, m, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$ is Frobenius according to Definition 5.4 if and only if either of the morphisms $\Phi_1, \Phi_2 \in \mathrm{Hom}(A, A^\vee)$ in (5.8) and (5.9) is invertible.

We will also be interested in modules and bimodules of an algebra A in $\mathcal{C}$. Their definition is again analogous to the classical case.

Definition 5.6:
Let $\mathcal{C}$ be a strict tensor category and A an algebra in $\mathcal{C}$.

(i) A *left module* over A is a pair $M=(\dot{M},\rho)$ consisting of an object $\dot{M}$ of $\mathcal{C}$ and a *representation morphism* $\rho\equiv\rho_M\in\mathrm{Hom}(A\otimes\dot{M},\dot{M})$, satisfying

$$\rho\circ(m\otimes id_{\dot{M}})=\rho\circ(id_A\otimes\rho) \qquad \text{and} \qquad \rho\circ(\eta\otimes id_{\dot{M}})=id_{\dot{M}}\,. \tag{5.10}$$

A *right module* over A is defined analogously.

(ii) An A-*bimodule* is a triple $M=(\dot{M},\rho_{\rm l},\rho_{\rm r})$ such that $(\dot{M},\rho_{\rm l})$ is a left A-module, $(\dot{M},\rho_{\rm r})$ is a right A-module, and the left and right actions of A commute.

For morphisms between A-modules that intertwine the module action we use the notation

$$\mathrm{Hom}_A(N,M):=\{f\in\mathrm{Hom}(\dot{N},\dot{M})\,|\,f\circ\rho_N=\rho_M\circ(id_A\otimes f)\}\ \ . \tag{5.11}$$

Similarly, the space $\mathrm{Hom}_{A|A}(N,M)$ for two A-bimodules N,M is the subspace of $\mathrm{Hom}(\dot{N},\dot{M})$ which intertwines both left and right action of A. Denote by $\mathcal{C}_A$ the category whose objects are left A-modules and whose morphism sets are given by Hom_A. Analogously, ${}_A\mathcal{C}_A$ denotes the category of A-bimodules with morphisms $\mathrm{Hom}_{A|A}$. Note that in contrast to $\mathcal{C}_A$, ${}_A\mathcal{C}_A$ is always a tensor category. The notion of an A-bimodule is also used to define when an algebra is simple.

Definition 5.7:
An algebra A in a tensor category $\mathcal{C}$ is called *simple* iff $\mathrm{Hom}_{A|A}(A,A)=\Bbbk\,id_A$, i.e. it is simple as a bimodule over itself.

The category $\mathcal{C}_A$ of left A-modules carries more structure than just that of a category. Note that if $M=(\dot{M},\rho)$ is a left A-module, then so is $(\dot{M}\otimes U,\rho\otimes id_U)$, for any object U of $\mathcal{C}$. This turns $\mathcal{C}_A$ into a (right) *module category* over $\mathcal{C}$, i.e. we have a bifunctor $\otimes\,:\mathcal{C}_A\times\mathcal{C}\to\mathcal{C}_A$ as well as an appropriate associator and unit constraints. The concept of a module category was introduced in [Pa2]; for more details and references see sections I:4.1 and IV:2.

Two algebras A and B in $\mathcal{C}$ are called *Morita-equivalent* if $\mathcal{C}_A$ and $\mathcal{C}_B$ are equivalent as module categories over $\mathcal{C}$ [Pa4, Pa5]. Morita equivalence is a much weaker notion than isomorphy of algebras.

5.2 New phenomenon in the braided setting

In the application to CFT we have in mind, the categories of interest are modular tensor categories. Let us in this chapter be more general and take $\mathcal{C}$ to be a ribbon category. We would like to investigate the special features that arise due to the presence of a braiding.

First note that already the notion of commutativity of an algebra requires a braiding. The braiding also allows to define the opposite algebra as well as the tensor product of two algebras.

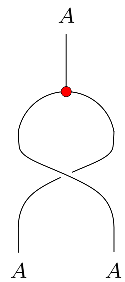
Definition 5.8:
Let $\mathcal{C}$ be a strict braided tensor category and let $A=(A,m,\eta)$ be an algebra in $\mathcal{C}$.

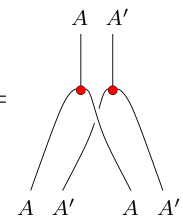
(i) A is said to be *commutative* iff $m\circ c_{A,A}=m$, or, equivalently, iff $m\circ c_{A,A}^{-1}=m$.

(ii) The *opposite algebra* of A is $A_{\rm op}=(A,m\circ c_{A,A}^{-1},\eta)$.

(iii) Given another algebra $A'=(A',m',\eta')$, the *tensor product algebra* $A\otimes A'$ has multiplication $m_{A\otimes A'}=(m\otimes m')\circ(id_A\otimes(c_{A,A'})^{-1}\otimes id_{A'})$ and unit $\eta_{A\otimes A'}:=\eta\otimes\eta'$.

In the pictorial notation, the multiplications m_{op} and $m_{A\otimes A'}$ take the form

$m_{\text{op}} :=$ 

$m_{A\otimes A'} :=$ 

(5.12)

A proof that the opposite algebra and the tensor product algebra are indeed algebras is given in Propositions I:3.18 and I:3.22.

Remark 5.9:
(i) In the definition of A_{op} and $A \otimes A'$ the inverse braiding has been used. In the first case one can in fact define a whole family of algebras $A^{(n)} = (A, m \circ (c_{A,A})^n, \eta)$ with $A_{\text{op}} = A^{(-1)}$. If $\mathcal{C}$ is in addition ribbon, then the twist θ_A provides an isomorphism $A^{(n)} \cong A^{(n+2)}$, cf. Proposition I:3.20. Second, instead of $A \otimes A'$ one can define $A \tilde{\otimes} A'$ which has the same unit as $A \otimes A'$, but the multiplication is now $m_{A \tilde{\otimes} A'} = (m \otimes m') \circ (id_A \otimes c_{A',A} \otimes id_{A'})$, see also remark I:3.23 (i). We will only work with the tensor product as given in Definition 5.8 (iii).

(ii) If A and A' are Frobenius then so are A_{op} and $A \otimes A'$ if we set $\varepsilon_{\text{op}} = \varepsilon$ and $\Delta_{\text{op}} = c_{A,A} \circ \Delta$, as well as

$$\varepsilon_{A \otimes A'} = \varepsilon_A \otimes \varepsilon_{A'} \qquad \text{and} \qquad \Delta_{A \otimes A'} = (id_A \otimes c_{A,A'} \otimes id_{A'}) \circ (\Delta_A \otimes \Delta_{A'}) \,. \tag{5.13}$$

If A and A' are special and/or symmetric, then so are A_{op} and $A \otimes A'$, see Propositions I:3.18 and I:3.22.

In the investigation of CFT on unoriented surfaces one needs the notion of a reversion on an algebra A [II]. It is defined as follows (Definition II:2.1).

Definition 5.10:
Let $\mathcal{C}$ be a strict ribbon category.

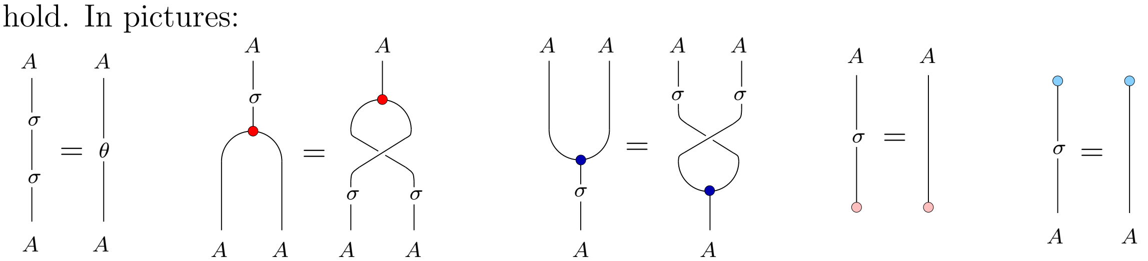
(i) A *reversion* on an algebra $A = (A, m, \eta)$ in $\mathcal{C}$ is an endomorphism $\sigma \in \text{Hom}(A, A)$ that is an algebra anti-homomorphism and squares to the twist, i.e.

$$\sigma \circ \eta = \eta \,, \qquad \sigma \circ m = m \circ c_{A,A} \circ (\sigma \otimes \sigma) \,, \qquad \sigma \circ \sigma = \theta_A \,. \tag{5.14}$$

If the algebra A is also a coalgebra, $A = (A, m, \eta, \Delta, \varepsilon)$, then we demand that in addition

$$\varepsilon \circ \sigma = \varepsilon \qquad \text{and} \qquad \Delta \circ \sigma = (\sigma \otimes \sigma) \circ c_{A,A} \circ \Delta \tag{5.15}$$

hold. In pictures:



(5.16)

(ii) The quadruple $A=(A,m,\eta,\sigma)$ consisting of an algebra and a reversion is called an *algebra with reversion*.

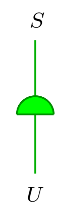
(iii) A symmetric special Frobenius algebra with reversion will also be called a *Jandl algebra*.

Remark 5.11:
In the classical case $\mathcal{C} = \mathcal{V}ect_f(\mathbb{k})$, an algebra with reversion is the same as an algebra with involution. However, for a ribbon category both requirements $\sigma \circ \sigma = id_A$ and $\sigma \circ \sigma = \theta_A$ are natural. The former is a direct generalisation of an involution, hence a different name was adopted for the latter.

In the treatment of algebras, the first significant difference to the classical case arises when one defines the centre of an algebra A: in the braided setting one has to consider two different centres [VZ, Os]. Here we will define left/right centres for symmetric special Frobenius algebras in terms of idempotents, which makes it applicable to a larger class of categories, cf. the discussion in section C:2.4.

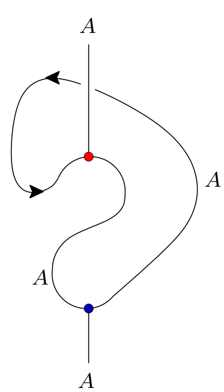
An *idempotent* is an endomorphism p such that $p \circ p = p$. A *retract* of an object U is a triple (S,e,r) with $e \in \mathrm{Hom}(S,U)$ (the embedding) and $r \in \mathrm{Hom}(U,S)$ (the restriction) such that $r \circ e = id_S$. An idempotent $p \in \mathrm{Hom}(U,U)$ is called *split* if there exists a retract (S,e,r) of U with $e \circ r = p$; in this case the retract (S,e,r) is unique up to isomorphism of retracts. We will use the following graphical notation for the embedding and restriction morphisms of a retract,

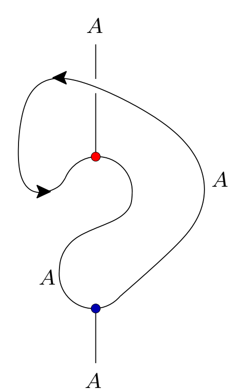
$e =$ 

$r =$ 

(5.17)

For the remainder of this section, let $\mathcal{C}$ be a strict ribbon category and A a symmetric special Frobenius algebra in $\mathcal{C}$. The following two morphisms are idempotents in $\mathrm{Hom}(A,A)$ (see section C:2.4):

$P_A^l :=$ 

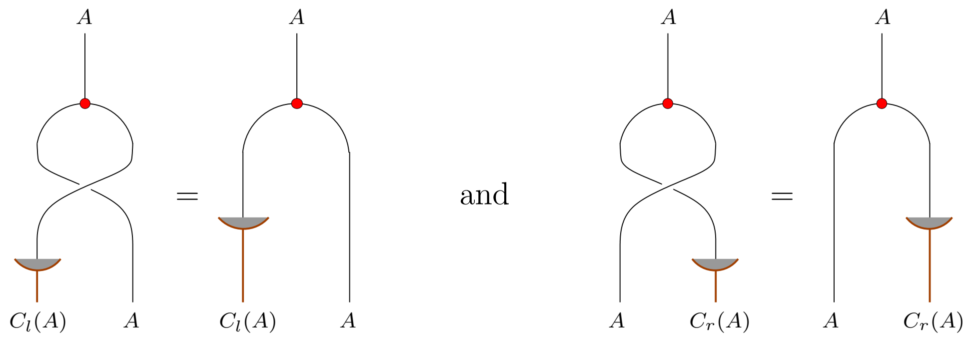
and $P_A^r :=$ 

(5.18)

Definition 5.12:
If the idempotent P_A^l is split, then a *left centre* of A is a retract $C_l(A) = (C_l(A), e_l, r_l)$ of A s.t. $e_l \circ r_l = P_A^l$. If the idempotent P_A^r is split, then a *right centre* of A is a retract $C_r(A) = (C_r(A), e_r, r_r)$ of A s.t. $e_r \circ r_r = P_A^r$.

Both, left and right centre are unique up to isomorphism of retracts. The centres $C_{l/r}(A)$ exist in particular if $\mathcal{C}$ is Karoubian, as then by definition every idempotent is split. All abelian categories, as well as all additive semisimple categories are Karoubian.

The left and right centres deserve their names because by remark C:2.34 we have



(5.19)

Conversely, by lemma C:2.32 $C_{l/r}(A)$ are maximal in the sense that any other retract of A with properties (5.19) factors through $C_{l/r}(A)$.

Since in the classical setting, i.e. in the category $\mathcal{V}ect_f(\Bbbk)$, the braiding is symmetric $c_{U,V} = c_{V,U}^{-1}$, it is easy to see from (5.18) that always $P_A^l = P_A^r$, s.t. left and right centre coincide. To construct a simple example where this is not the case, we can endow the category of graded vector spaces with a non-symmetric braiding.

Example 5.13:
Let $\Bbbk = \mathbb{C}$, or alternatively any field which contains an element (-1) with the property $(-1)^2 = 1$. Consider the category $\mathcal{C}(G, 1, \Omega)$ for the group $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. Since $\psi \equiv 1$, the associator in $\mathcal{C}(G, 1, \Omega)$ is the same as in the category of G-graded vector spaces. We will, however, choose a non-standard braiding by setting, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\Omega(x, y) = (-1)^{(x_1+x_2)y_2} \in \Bbbk$. For $(1, \Omega)$ to be an abelian 3-cocycle, Ω has to be a bihomomorphism (see (III:A.8)), which is easily verified.
Since $\psi \equiv 1$, the group algebra $A = \Bbbk[G]$ is also an algebra in $\mathcal{C}(G, 1, \Omega)$. One can verify that choosing $\varepsilon : A \to \Bbbk$, $\varepsilon(g) = 8\,\delta_{g,e}$ with e the unit of G, turns A into a symmetric special Frobenius algebra (recall that the definition of symmetric did not involve the braiding; the factor of 8 accounts for the normalisation convention $\varepsilon \circ \eta = \dim(A)$, cf. the text after Theorem 2.3).
Combining (3.14) and (5.19) we see that the left centre is given by (compare also to Proposition III:3.29 and eqn. (III:3.38))

$$\begin{aligned} C_l(A) &= \operatorname{span}_{\Bbbk}\big\{ c \in G \;\big|\; \Omega(a, c)^{-1} = 1 \;\forall a \in G \,\big\} \\ &= \operatorname{span}_{\Bbbk}\big\{ (c_1, c_2) \in G \;\big|\; (a_1{+}a_2)c_2 \equiv 0 \bmod 2 \;\forall (a_1, a_2) \in G \,\big\} \\ &= \operatorname{span}_{\Bbbk}\big\{ (0,0), (0,2), (1,0), (1,2) \big\} \;\;\cong\;\; \Bbbk[\,\mathbb{Z}_2 \times \mathbb{Z}_2\,] \;\; . \end{aligned} \tag{5.20}$$

In the same way, for the right centre we find

$$\begin{aligned} C_r(A) &= \operatorname{span}_{\Bbbk}\big\{ c \in G \;\big|\; \Omega(c, a)^{-1} = 1 \;\forall a \in G \,\big\} \\ &= \operatorname{span}_{\Bbbk}\big\{ (0,0), (0,2), (1,1), (1,3) \big\} \;\;\cong\;\; \Bbbk[\mathbb{Z}_4] \;\; . \end{aligned} \tag{5.21}$$

Thus $C_l(A)$ and $C_r(A)$ are distinct subobjects of A, and they are not isomorphic as algebras. In fact, as we will see now, they are not even Morita-equivalent.
Using the method of induced modules (see sections I:4.3 and III:4.2 for more details and references) one computes that $\mathcal{C}_{C_l(A)}$ has two simple objects, $M_1^l = C_l(A)$ and $M_2^l = C_l(A) \otimes L_{(0,1)}$. As objects in $\mathcal{C}$ we have (L_g denotes the simple object $\Bbbk g$ in $\mathcal{C}(G,1,\Omega)$)

$$\dot{M}_1^l \cong L_{(0,0)} \oplus L_{(0,2)} \oplus L_{(1,0)} \oplus L_{(1,2)} \quad , \qquad \dot{M}_2^l \cong L_{(0,1)} \oplus L_{(0,3)} \oplus L_{(1,1)} \oplus L_{(1,3)} \tag{5.22}$$

Similarly, for $C_r(A)$ we find two simple modules $M_1^r = C_r(A)$ and $M_2^r = C_r(A) \otimes L_{(0,1)}$, which as objects in $\mathcal{C}$ are given by

$$\dot{M}_1^r \cong L_{(0,0)} \oplus L_{(0,2)} \oplus L_{(1,1)} \oplus L_{(1,3)} \quad , \qquad \dot{M}_2^r \cong L_{(0,1)} \oplus L_{(0,3)} \oplus L_{(1,2)} \oplus L_{(1,0)} \tag{5.23}$$

For the action of $L_{(1,0)} \in \mathrm{Obj}(\mathcal{C})$ on $\mathcal{C}_{C_l(A)}$ and $\mathcal{C}_{C_r(A)}$ we find

$$\begin{array}{lll} M_1^l \otimes L_{(1,0)} \cong M_1^l & , \qquad & M_1^r \otimes L_{(1,0)} \cong M_2^r \\ M_2^l \otimes L_{(1,0)} \cong M_2^l & , \qquad & M_2^r \otimes L_{(1,0)} \cong M_1^r \end{array} \tag{5.24}$$

so that $\mathcal{C}_{C_l(A)}$ and $\mathcal{C}_{C_r(A)}$ are not equivalent as module categories over $\mathcal{C}$.

The left and right centre of A inherit natural structure as a retract of A.

Proposition 5.14:
Let $\mathcal{C}$ be a strict ribbon category and A a symmetric special Frobenius algebra in $\mathcal{C}$ such that $C_l(A)$ and $C_r(A)$ exist.

(i) $C_l(A)$ and $C_r(A)$ are commutative symmetric Frobenius algebras.

(ii) If A is simple and $\dim C_{l/r}(A) \neq 0$, then $C_l(A)$ and $C_r(A)$ are simple and special.

This is shown in Proposition C:2.37; the multiplication and comultiplication on $C_{l/r}(A)$ are given explicitly in (C:2.70).

Frobenius algebras in $\mathcal{R}ep\,\widehat{su}(2)_k$

As an example for commutative Frobenius algebras consider the category $\mathcal{C} = \mathcal{R}ep\,\widehat{su}(2)_k$ of integrable representations of $\widehat{su}(2)_k$, or, equivalently, a semi-simple quotient of the category of finite-dimensional representations of $U_q(sl(2))$ at the root of unity $q = e^{\pi i/(k+2)}$. There is the following classification theorem [KO, Theorem 6.1].

Theorem 5.15:
There is a one-to-one correspondence between commutative simple symmetric special Frobenius algebras in $\mathcal{C}$ and Dynkin diagrams of type A_n, D_{2n}, E_6 and E_8. Algebras of type A_n occur at all levels $k \in \mathbb{Z}_{\geq 0}$, algebras of type D_{2n} for $k \cong 0 \bmod 4$, algebras of type E_6 and E_8 at k=10 and for k=28, respectively.

Remark 5.16:
(i) This theorem can be formulated in the framework of operator algebra and subfactors. In fact, it is in this context that it has first been stated and proved [Oc, Pp, BN1, BN2, I1, I2].

(ii) The classification given in the theorem above should be compared to that of Theorem 4.2. One finds an almost complete agreement, except that the cases $D_{\rm odd}$ and E_7 are missing. The agreement is prefect if one instead classifies module categories over $\mathcal{C} = \mathcal{R}ep\,\widehat{su}(2)_k$. This is done in [Os, Theorem 6], see also [BEK2] for related subfactor results.

In view of this remark, the relevant objects to consider for comparision to CFT are module categories. This finding is explained by combining the construction of a consistent set of CFT correlators from a symmetric special Frobenius algebra in section 6 with the following theorem (obtained as a special case of [Os, Theorem 1] and [Os, Theorem 3])

Theorem 5.17:
Let $\mathcal{C}$ be a modular tensor category and let $\mathcal{M}$ be a semisimple indecomposable module category over $\mathcal{C}$. Then there exists a symmetric special Frobenius algebra A in $\mathcal{C}$ with $\dim \mathrm{Hom}(A, \mathbf{1}) = 1$ such that the module categories $\mathcal{M}$ and $\mathcal{C}_A$ are equivalent.

Several symmetric special Frobenius algebras A can give rise to the same module category $\mathcal{C}_A$. By definition these algebras are then Morita equivalent. In the construction of CFT correlators in chapter 6 one finds that Morita equivalent algebras lead to equivalent CFTs, cf. Remark 6.12 (iv). One of the merits of the construction in chapter 6 is, that it explains the correspondence between module categories and torus partition functions found 'experimentally' for $su(2)$: a module category defines a Morita class of symmetric special Frobenius algebras, and by Theorem 6.11 below each such algebra defines a consistent set of correlators. Evaluating the correlators on the torus, one recovers the result of Theorem 4.2. In particular, module categories give rise to consistent CFTs, not only to modular invariant bilinear combinations of characters.

5.3 Local modules

Let A be a commutative symmetric special Frobenius algebra in a strict ribbon category $\mathcal{C}$. Another property which is not found in symmetric tensor categories is the presence of an interesting subclass of A-modules, the so-called local [KO], or dyslectic [Pa6], A-modules, see the discussion below Definition C:3.15 and Proposition C:3.17 for the relation between these definitions.

Definition 5.18:
Let A be a commutative symmetric special Frobenius algebra in a ribbon category $\mathcal{C}$. A left A-module M is called *local* iff $\theta_M \in \mathrm{Hom}_A(M, M)$.

Analogous to the category $\mathcal{C}_A$ of all A-modules, one can define the full subcategory $\mathcal{C}_A^{\ell oc}$ of all local A-modules. The importance of the category $\mathcal{C}_A^{\ell oc}$ lies in the fact that it inherits many of the properties of $\mathcal{C}$, which is in general not so for $\mathcal{C}_A$.

Proposition 5.19:
Let $\mathcal{C}$ be an additive, $\Bbbk$-linear, strict ribbon category $\mathcal{C}$ which is also Karoubian, and the tensor unit $\mathbf{1}\in\mathrm{Obj}(\mathcal{C})$ is simple as well as absolutely simple. For every commutative symmetric special Frobenius algebra A in $\mathcal{C}$ the following holds:

(i) $\mathcal{C}_A^{\ell oc}$ is a ribbon category.

(ii) If $\mathcal{C}$ is semisimple, then $\mathcal{C}_A^{\ell oc}$ is semisimple. If $\mathcal{C}$ is closed under direct sums and subobjects, then $\mathcal{C}_A^{\ell oc}$ is closed under direct sums and subobjects.

(iii) If $\mathcal{C}$ is modular and if A is in addition simple, then $\mathcal{C}_A^{\ell oc}$ is modular.

The proof is a combination of the results in [Pa6, KO, FuS], see Proposition C:3.21. We can now state the first main result of [C], namely Theorem C:5.20.

Theorem 5.20:
Let $\mathcal{C}$ be an additive, $\Bbbk$-linear, strict ribbon category $\mathcal{C}$ which is also Karoubian, and the tensor unit $\mathbf{1}\in\mathrm{Obj}(\mathcal{C})$ is simple as well as absolutely simple. Let A be a symmetric special Frobenius algebra in $\mathcal{C}$ such that the symmetric Frobenius algebras $C_{l/r}(A)$ are special as well. Then there is an equivalence

$$\mathcal{C}^{\ell oc}_{C_l(A)} \;\cong\; \mathcal{C}^{\ell oc}_{C_r(A)} \tag{5.25}$$

of ribbon categories.

Remark 5.21:
(i) The equivalence of the categories of local modules over the left and right centres given in Theorem 5.20 is a category theoretic analogue of Theorem 5.5 of [BE1], which was obtained in the study of relations between nets of braided subfactors and modular invariants. In the context of module categories, this equivalence has been formulated, as a conjecture (claim 5), in [Os, section 5.4].
The importance of the category theoretic result Theorem 5.20 is, that it does not need on an underlying realisation of the category $\mathcal{C}$ via subfactors. In particular, in the subfactor language one always takes $\Bbbk = \mathbb{C}$, and one assumes the existence of a conjugation. Theorem 5.20 thus has a wider range of applicability.

(iii) In [MS2] it was shown that the modular invariant partition function of a rational CFT takes the form of a 'fusion rule automorphism on top of maximal extensions of the chiral algebra'. This statement can be recovered from Theorem 5.20, together with the construction of CFT correlators from symmetric special Frobenius algebras as in section 6. For more details see remark C:5.24 (ii).

Theorem 5.20 has no classical analogue. For algebras over a field $\Bbbk$, the left and right centre coincide, and the statement is trivial. In the genuinely braided case $C_l(A)$ and $C_r(A)$ can be distinct subobjects, non-isomorphic as algebras and even non-Morita equivalent, as we saw explicitly in example 5.13.

Example 5.22:
Continuing example 5.13, recall that the simple $C_l(A)$-modules where given by (5.22) and the simple $C_r(A)$-modules by (5.23). Let us apply Definition 5.18 to see which of these are local. By (3.14), the twist on L_g is given by multiplication with $\Omega(g,g)^{-1}$. Thus on $L_{(0,0)}$, $L_{(0,2)}$, $L_{(1,0)}$, $L_{(1,2)}$, $L_{(1,1)}$ and $L_{(1,3)}$ the twist is just the identity, while on $L_{(0,1)}$ and $L_{(0,3)}$ it acts by multiplication with -1.
A simple module is local iff the twist is a multiple of the identity morphism (Corollary C:3.18). From the explicit decompositions (5.22) and (5.23) we see that the only simple object in $\mathcal{C}^{\ell oc}_{C_l(A)}$ is $C_l(A)$ and the only simple object in $\mathcal{C}^{\ell oc}_{C_r(A)}$ is $C_r(A)$. In particular, both categories are equivalent as implied by Theorem 5.20.

In order to state the second main result of [C] we need two more notions. First, given two $\Bbbk$-linear categories $\mathcal{C}$ and $\mathcal{D}$, their *Karoubian product* $\mathcal{C} \boxtimes \mathcal{D}$ is obtained in two steps. One starts with category whose objects are pairs $U \times V$ with $U \in \mathrm{Obj}(\mathcal{C})$ and $V \in \mathrm{Obj}(\mathcal{D})$, and whose morphism sets $U \times V \to X \times Y$ are tensor products $\mathrm{Hom}_{\mathcal{C}}(U,X) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{D}}(V,Y)$. Then one takes the Karoubian envelope, i.e. one completes the category so obtained with respect to idempotents, cf. definitions C:2.7 and C:6.1. Second, the *dual category* $\overline{\mathcal{C}}$ of a braided tensor category $(\mathcal{C}, \otimes, c)$ is the braided tensor category $(\mathcal{C}^{\mathrm{opp}}, \otimes, c^{-1})$, see definition C:6.13. We can now formulate

Theorem 5.23:
Let $\mathcal{Q}$, $\mathcal{H}$ be modular tensor categories. Let A be a commutative symmetric special Frobenius algebra in $\mathcal{Q} \boxtimes \mathcal{H}$ s.t. the only subobject of A of the form $U_{\mathcal{Q}} \times \mathbf{1}_{\mathcal{H}}$ is $\mathbf{1}_{\mathcal{Q}} \times \mathbf{1}_{\mathcal{H}}$. Denote by $\mathcal{G}$ the modular tensor category

$$\mathcal{G} = \big(\mathcal{Q} \boxtimes \mathcal{H} \big)^{\ell oc}_{A} \quad . \tag{5.26}$$

Then there exists a commutative symmetric special Frobenius algebra B in $\mathcal{G} \boxtimes \overline{\mathcal{H}}$ s.t. the only subobject of A of the form $U_{\mathcal{Q}} \times \mathbf{1}_{\mathcal{H}}$ is $\mathbf{1}_{\mathcal{Q}} \times \mathbf{1}_{\mathcal{H}}$ and s.t.

$$\mathcal{Q} = \big(\mathcal{G} \boxtimes \overline{\mathcal{H}} \big)^{\ell oc}_{B} \quad . \tag{5.27}$$

This theorem is proved under slightly weaker assumptions in Theorem C:7.6, where also the algebra B is constructed. The basic idea of the proof is to find an algebra F in $\mathcal{D} = \mathcal{Q} \boxtimes \mathcal{H} \boxtimes \overline{\mathcal{H}}$ with the properties $\mathcal{D}^{\ell oc}_{C_l(F)} \cong \mathcal{Q}$ and $\mathcal{D}^{\ell oc}_{C_r(F)} \cong (\mathcal{G} \boxtimes \overline{\mathcal{H}})^{\ell oc}_B$. The statement (5.27) then follows from Theorem 5.20.

Note that also Theorem 5.23 does not have a classical analogue. If applied to categories of vector spaces, the constraints imposed on A would force $A \cong \mathbf{1}$. It shows, however, that if the braiding is ‘maximally non-symmetric’ a problem analogous to the one posed in section 1.3 does have a solution. The formulation of Theorem 5.23 is motivated by the so-called coset construction in chiral conformal field theory (see e.g. [DMS, chapter 18]), for more details on this relation refer to [FFRS2].

5.4 α-induced bimodules

Let A be an algebra in a braided tensor category $\mathcal{C}$ The presence of a braiding allows us to define two tensor functors $\alpha^{\pm}_A$ from $\mathcal{C}$ to ${}_A\mathcal{C}_A$, called *α-induction*. To an object $V \in \mathrm{Obj}(\mathcal{C})$ we assign the A-bimodule

$$\alpha^{\pm}_A(V) := \big(A \otimes V,\, m \otimes id_V,\, \rho^{\pm}_{\mathrm{r}} \big) \tag{5.28}$$

where the right representation morphisms $\rho_{\rm r}^{\pm} \in \mathrm{Hom}(A\otimes V\otimes A, A\otimes V)$ are

$$\rho_{\rm r}^{+} := (m\otimes id_V)\circ(id_A\otimes c_{V,A}) \qquad \text{and} \qquad \rho_{\rm r}^{-} := (m\otimes id_V)\circ(id_A\otimes (c_{A,V})^{-1})\,, \tag{5.29}$$

respectively (for a figure see (6.11) below). On a morphism $f \in \mathrm{Hom}(V,W)$, $\alpha_A^{\pm}$ acts as

$$\alpha_A^{\pm}(f) := id_A\otimes f \;\in \mathrm{Hom}(A{\otimes}V, A{\otimes}W) \;\;. \tag{5.30}$$

One can verify that the α-inductions $\alpha_A^{\pm}$ are indeed tensor functors. They were first studied in the theory of subfactors (see [LRe1] and also e.g. [X, BE3, BEK3]), and were reformulated in the form used here in [Os].

Define also the non-negative integers (cf. section I:5.4 and remark C:2.28; these numbers where first introduced in the context of subfactor theory [BEK1, Definition 5.5])

$$\tilde Z(A)_{U,V} := \dim_{\Bbbk}\big[\mathrm{Hom}_{A|A}(\alpha_A^{-}(V),\alpha_A^{+}(U))\big]\,. \tag{5.31}$$

Suppose now that $\mathcal{C}$ is a modular tensor category. Then there is a finite label set $\mathcal{I}$ for simple objects U_i and all integers $\tilde Z(A)_{U,V}$ can be recovered from the corresponding expression for simple objects, $\tilde Z(A)_{ij} \equiv \tilde Z(A)_{U_i,U_j}$. We will write $\tilde Z(A)$ for the $\mathcal{I}\times\mathcal{I}$-matrix with entries $\tilde Z(A)_{ij}$. This matrix has a number of surprising properties.

Theorem 5.24:
Let A be a symmetric special Frobenius algebra in a modular tensor category $\mathcal{C}$. Then
(i) as objects in $\mathcal{C}$ the left and right centre of A are isomorphic to

$$C_l(A) \cong \bigoplus_{i\in\mathcal{I}} U_i^{\oplus \tilde Z(A)_{i0}} \qquad \text{and} \qquad C_r(A) \cong \bigoplus_{j\in\mathcal{I}} U_j^{\oplus \tilde Z(A)_{0j}} \tag{5.32}$$

(ii) $\tilde Z(A_{\rm op}) = \tilde Z(A)^{\rm t}$.
(iii) given a second symmetric special Frobenius algebra B in $\mathcal{C}$, we have $\tilde Z(A\otimes B) = \tilde Z(A)\tilde Z(B)$, as matrix product.
(iv) the number of isomorphism classes of simple objects in $\mathcal{C}_A$ is given by $\mathrm{tr}\,\tilde Z(A)$.
(v) $\big[\hat S, \tilde Z(A)\big] = 0$ and $\big[\hat T, \tilde Z(A)\big] = 0$ for $\hat S$ and $\hat T$ as given in (3.27).
(vi) $\tilde Z(A)_{ij} = \tilde Z(A)_{\bar\imath\bar\jmath}$.
(vii) if A is simple, then $\tilde Z(A)_{ij} \le \dim(U_i)\,\dim(U_j)$.

The proof of part (i) is obtained from Lemma C:3.13, Proposition C:3.6 and Remark C:3.7. The proof of parts (ii) and (iii) is given in Proposition I:5.3, part (iv) is demonstrated in Theorem I:5.18, part (v) amounts to Theorem I:5.1 (i), part (vi) follows from Theorem I:5.23 (ii,iii) and finally part (vii) is proved in [BE2, section 1]. An alternative proof for (vii) is given in Lemma III:3.5.

Remark 5.25:
(i) For a symmetric special Frobenius algebra A over $\Bbbk$, $Z(A) \equiv Z(A)_{00}$ is just the dimension of the centre of A. (This follows from Theorem I:5.1 together with the fact that for $\mathcal{C} = \mathcal{V}ect_f(\Bbbk)$

we have $A_{\mathrm{top}} \cong A$.) It is intriguing that when considering arbitrary modular tensor categories, the dimension of the centre of an algebra generalises to a matrix.

(ii) Property (iv) is also useful to compute the number simple of A-bimodules. To this end, one notes that A-bimodules are in one-to-one correspondence to $A \otimes A_{\mathrm{op}}$-left modules. For the latter, by points (ii) and (iii) of the theorem, the number of irreducible modules is given by $\mathrm{tr}\big(\tilde{Z}(A)\tilde{Z}(A)^{\mathrm{t}}\big)$, cf. Remark I:5.19 (ii).

(iii) Recall that a modular tensor category gives rise to a 3dTFT. The numbers $\tilde{Z}(A)_{ij}$ can be computed as invariants of ribbon graphs in $S^2{\times}S^1$. The corresponding ribbon graph is obtained from the general construction of CFT correlators in section 6, applied to the torus. The resulting graph is given in (I:5.30) and the relation to (5.31) is provided in section I:5.4. In accordance with (i) above and the discussion in the end of section 2.3, when applied to the category of vector spaces one obtains a 2d lattice TFT and its correlator on the torus is given by the dimension of the centre of A.

6 From Algebras to 2dCFT

In this chapter we will combine the tools described in chapters 3 and 5 to construct CFT correlators. As discussed in section 1.2, this will be done on the algebraic level, not on the complex-analytic level. In particular, also here we will work with an arbitrary field $\Bbbk$, not necessarily with the complex numbers.

We will start with a precise statement of the problem we want to solve, and then describe how to construct a solution starting from a symmetric special Frobenius algebra in a tensor category.

6.1 Statement of problem

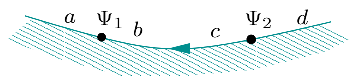
Definition 6.1:
Let $\mathcal{C}$ be a modular tensor category with a field $\Bbbk$ as ground ring, and let $\{\,U_i\,|\,i{\in}\mathcal{I}\,\}$ be representatives of the simple objects.

(i) A *choice of field data* consists of
- a finite dimensional $\Bbbk$-vector space ϕ_{ij} for each pair $i,j \in \mathcal{I}$, the *bulk field degeneracy spaces*
- a set $\mathcal{B}$, the *set of boundary conditions*
- for each $k \in \mathcal{I}$ and each pair $a,b \in \mathcal{B}$, a finite dimensional $\Bbbk$-vector space $\psi_{k,ab}$, the *boundary field degeneracy spaces.*

(ii) A *topological world sheet* X is a compact, two-dimensional manifold, also denoted by X (which may have non-empty boundary and may be non-orientable), together with a finite, unordered set of marked points and an orientation or$(\partial$X) of its boundary. A marked point is either
- a *bulk insertion*, that is, a tupel $\Phi = (i,j,\phi,p,[\gamma],\mathrm{or}_2(p))$, where $i,j \in \mathcal{I}$, $\phi \in \phi_{ij}$, $p \in \mathrm{X}\backslash\partial\mathrm{X}$, $[\gamma]$ is an arc-germ with $\gamma(0){=}p$ and $\mathrm{or}_2(p)$ is an orientation of a neighbourhood of $p \in$ X.
- a *boundary insertion*, that is, a tupel $\Psi = (a,b,k,\psi,p,[\gamma])$ where $a,b \in \mathcal{B}$, $k \in \mathcal{I}$, $\psi \in \psi_{k,ab}$, $p \in \partial$X and $[\gamma]$ is an arc-germ with $\gamma(0){=}p$. There has to be a representative γ of $[\gamma]$ which is

a subset of ∂X.

No two bulk or boundary insertions are allowed to be at the same point of X. If two boundary insertions $\Psi_1 = (a, b, k, \psi_1, p_1, [\gamma_1])$ and $\Psi_2 = (c, d, l, \psi_2, p_2, [\gamma_2])$ are adjacent and Ψ_1 is “after” Ψ_2 w.r.t. or(∂X),



(6.1)

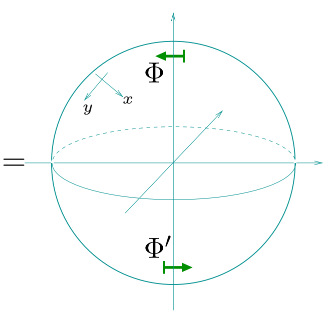
then we require $b = c$. If there is a connected component of ∂X without boundary insertions, it gets labelled by an element of $\mathcal{B}$.

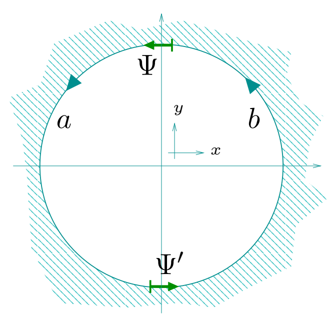
(iii) An *oriented topological world sheet* X is topological world sheet X together with an orientation $\mathrm{or}_2(\mathrm{X})$ of X. For a bulk insertion $\Phi = (i, j, \phi, p, [\gamma], \mathrm{or}_2(p))$, $\mathrm{or}_2(p)$ is required to agree with $\mathrm{or}_2(\mathrm{X})$. Also, $\mathrm{or}_2(\mathrm{X})$ induces an orientation of ∂X via the inward pointing normal, which is required to agree with or(∂X).

Remark 6.2:

(i) If $\mathfrak{V}$ is a rational chiral algebra and $\mathcal{C} = \mathcal{R}ep(\mathfrak{V})$, then the space of bulk fields is given by (4.7) with $Z_{ij} = \dim_{\mathbb{C}} \phi_{ij}$. In particular, according to (4.19), the matrix Z so obtained should commute with S and T, or, equivalently, with $\hat{S}$ and $\hat{T}$ (see Remark 4.1).

(ii) Here are two examples of topological world sheets, a two-point function on the sphere and a two-point function on the disc

$\mathrm{X}_{S^2}(\Phi, \Phi') =$


$\mathrm{X}_D(\Psi, \Psi') =$


(6.2)

(iii) The topological world sheet is the analogue of the world sheet X^g as defined in sections 2.5 and 2.6. For example, the arc-germ entering the data of a marked point is what remains of the germ of local coordinates on a Riemannian world sheet.

The double of a topological world sheet is defined in complete analogy with the Riemannian case (4.9).

Definition 6.3:

The double $\hat{\mathrm{X}}$ of a topological world sheet X is the orientation bundle over X divided by an equivalence relation,

$$\hat{\mathrm{X}} = \mathrm{Or}(\mathrm{X})/\sim \qquad \text{with} \quad (x, \mathrm{or}_2) \sim (x, -\mathrm{or}_2) \quad \text{for} \quad x \in \partial\mathrm{X} \quad , \tag{6.3}$$

together with a finite collection of marked points and a choice of Lagrangian subspace of $H_1(\hat{\mathrm{X}},\mathbb{R})$. The marked points on $\hat{\mathrm{X}}$ are obtained as follows.

- Every boundary field insertion $\Psi=(a,b,k,\psi,p,[\gamma])$ on X gives rise to a marked point $(\tilde{p},[\tilde{\gamma}],U_k,+)$ on the double. Here $\tilde{p}=[p,\pm\mathrm{or}_2]$ and $[\tilde{\gamma}]$ is obtained by choosing a representative γ: $[-\delta,\delta]\to\partial\mathrm{X}$ of $[\gamma]$ which lies in $\partial\mathrm{X}$; a representative $\tilde{\gamma}$: $[-\delta,\delta]\to\hat{\mathrm{X}}$ of $[\tilde{\gamma}]$ is then given by $\tilde{\gamma}(t):=[\gamma(t),\pm\mathrm{or}_2]$ (compare to section IV:3.2).
- Every bulk field insertion $\Phi=\big(i,j,\phi,p,[\gamma],\mathrm{or}_2(p)\big)$ gives rise to two marked points $(\tilde{p}_i,[\tilde{\gamma}_i],U_i,+)$ and $(\tilde{p}_j,[\tilde{\gamma}_j],U_j,+)$ on the double $\hat{\mathrm{X}}$. Here $\tilde{p}_i=[p,\mathrm{or}_2(p)]$, $\tilde{p}_j=[p,-\mathrm{or}_2(p)]$, and similar for $\tilde{\gamma}_i(t)$ and $\tilde{\gamma}_j(t)$ (compare to section IV:3.3).

Finally, the relevant Lagrangian subspace of $H_1(\hat{\mathrm{X}},\mathbb{R})$ is given above remark II:3.1. The so-defined double $\hat{\mathrm{X}}$ is an extended surface.

An *isomorphism* $f:\mathrm{X}\to\mathrm{Y}$ *of topological world sheets* is defined similar to an isomorphism of extended surfaces (see section 3.6), i.e. it is a (degree one) homeomorphism f from X to Y compatible with the marked points as well as with orientation and labelling of the boundary components. An isomorphism of oriented topological world sheets is in addition orientation preserving.

An isomorphism $f:\mathrm{X}\to\mathrm{Y}$ induces an isomorphism $\hat{f}:\hat{\mathrm{X}}\to\hat{\mathrm{Y}}$ between the doubles of X and Y via

$$\hat{f}\big([x,\mathrm{or}]\big)=[f(x),f_*(\mathrm{or})] \quad , \tag{6.4}$$

where f_* denotes the push-forward of the local orientation or at $x\in\mathrm{X}$ via f. Recall that as an isomorphism of extended surfaces, $\hat{f}$ in turn induces an isomorphism $(\hat{f})_\sharp$ between state spaces of a 3dTFT, see section 3.6.

We have now gathered all the necessary notation to state the two problems to which we would like to find solutions.

Problem 6.4:
Given a modular tensor category $\mathcal{C}$, find a choice of field data $\{\phi_{ij},\mathcal{B},\psi_{k,ab}\}$ and an assignment

$$C \;:\; \mathrm{X}\longmapsto C(\mathrm{X})\in\mathcal{H}(\hat{\mathrm{X}}) \quad , \tag{6.5}$$

for any topological world sheet X. Here $\mathcal{H}(\hat{\mathrm{X}})$ is the $\Bbbk$-vector space assigned to the extended surface $\hat{\mathrm{X}}$ by the 3dTFT $(Z,\mathcal{H})$ constructed from $\mathcal{C}$. The assignment C must have the following five properties.

(i) (*Non-degeneracy of the bulk two-point function*) For every non-zero bulk insertion Φ there exists a bulk insertion Φ' such that $C\big(\mathrm{X}_{S^2}(\Phi,\Phi')\big)\neq 0$. The worldsheet $\mathrm{X}_{S^2}(\Phi,\Phi')$ is the one given in (6.2).

(ii) (*Non-degeneracy of the boundary two-point function*) For every non-zero boundary insertion Ψ there exists a boundary insertion Ψ' such that $C\big(\mathrm{X}_D(\Psi,\Psi')\big)\neq 0$. The worldsheet $\mathrm{X}_D(\Psi,\Psi')$ is given in (6.2).

(iii) (*Covariance under isomorphisms*) Let X, Y be two topological world sheets and $f:\mathrm{X}\to\mathrm{Y}$ an isomorphism of topological world sheets. Then $(\hat{f})_\sharp\circ C(\mathrm{X})=C(\mathrm{Y})$.

(iv) (*Factorisation of bulk correlators*) Let A_ε be an annulus of width 2ε as in section 2.4. For any continuous injection $u : A_\varepsilon \to \mathrm{X}$ s.t. the image of u does not contain any marked point of X we have

$$C(\mathrm{X}) = \sum_{\alpha,\beta} U^{\mathrm{bulk}}_{\alpha,\beta} \; G^{\mathrm{bulk}}_{u,\alpha\beta}\big(\, C(\Gamma^{\mathrm{bulk}}_{u,\alpha\beta}(\mathrm{X}))\,\big)\,, \tag{6.6}$$

(The various ingredients will be briefly explained below, for details see [FFRS].)

(v) (*Factorisation of boundary correlators*) Let $I_\varepsilon = [-1,1] \times [-\varepsilon,\varepsilon]$ for some $\varepsilon > 0$. For any continuous injection $u : I_\varepsilon \to \mathrm{X}$, s.t. the image of u does not contain any marked point of X, and s.t. the images of $\{-1\} \times [-\varepsilon,\varepsilon]$ and $\{1\} \times [-\varepsilon,\varepsilon]$ are subsets of $\partial\mathrm{X}$, we have

$$C(\mathrm{X}) = \sum_{\alpha,\beta} U^{\mathrm{bnd}}_{\alpha,\beta} \; G^{\mathrm{bnd}}_{u,\alpha\beta}\big(\, C(\Gamma^{\mathrm{bnd}}_{u,\alpha\beta}(\mathrm{X}))\,\big)\,. \tag{6.7}$$

(The same comments as in (iv) apply.)

Remark 6.5:

(i) Clearly, the factorisation conditions (iv), (v) are the most complicated of the above requirements. The precise statement of all ingredients is somewhat lengthy and can be found in [FFRS].

In words, $\Gamma^{\mathrm{bulk}}_{u,\alpha\beta}(\mathrm{X})$ is a new world sheet obtained from X by cutting X along the image of the unit circle under u and gluing half-spheres to the resulting holes. The two half-spheres carry bulk field insertions Φ_α and Φ_β, respectively. Here, $\{\Phi_\alpha\}$ denotes a basis of bulk fields and in the sum, α and β run over the index set for this basis. Denoting $\mathrm{X}' = \Gamma^{\mathrm{bulk}}_{u,\alpha\beta}(\mathrm{X})$, $G^{\mathrm{bulk}}_{u,\alpha\beta} : \mathcal{H}(\hat{\mathrm{X}}') \to \mathcal{H}(\hat{\mathrm{X}})$ is a linear map obtained by applying Z to a certain cobordism analogous to the one in the formulation of the factorisation property in section 3.6. The $U^{\mathrm{bulk}}_{\alpha,\beta} \in \Bbbk$ are related to the inverse of the two-point function on the sphere (which exists due to condition (i)). The meaning of the various objects in (v) is similar.

(ii) Condition (iii) above implies in particular invariance under the action of the mapping class group of the world sheet X, as can be seen by restricting (iii) to the case $\mathrm{X} = \mathrm{Y}$.

(iii) The factorisation conditions are the analogue of the factorisation of correlators in the complex-analytic formulation as sketched in (2.23).

In the formulation Problem 6.4, the world sheet X was allowed to be orientable or non-orientable. One can also formulate an analogous problem by restricting to oriented world sheets. This is consistent, because the applying the cutting procedure in conditions (iv), (v) to an oriented world sheet does again produce an oriented world sheet (while, on the other hand, it would not be consistent to restrict oneself to non-orientable world sheets).

Problem 6.6:

Solve the Problem 6.4 with the following modifications. The topological world sheet X in (6.5) is required to be oriented. The isomorphism f in (iii) has to be orientation preserving. The embeddings u in (iv) and (v) are required to be orientation preserving.

Clearly, solving Problem 6.4 in particular provides a solution for Problem 6.6. However, not every solution of 6.6 can be extended to a solution of 6.4. This is also displayed clearly in the construction of sections 6.3 and 6.4 below, where a symmetric special Frobenius algebra provides a solution for Problem 6.6, while according to [II], for Problem 6.4 we need a symmetric special Frobenius algebra with reversion, i.e. a Jandl algebra.

Let us call a solution $(\mathcal{U}, C)$ of Problem 6.4 a *set of correlators for the chiral data* $\mathcal{C}$ and a solution of Problem 6.6 a *set of oriented correlators for the chiral data* $\mathcal{C}$.

6.2 Appearance of Frobenius algebras

Below we will only treat Problem 6.6, the analogous steps for Problem 6.4 are explained in [II] and [IV], while the proof that the conditions (i)–(v) are satisfied is given in [FFRS].

The ansatz for C formulated below requires the specification of a symmetric special Frobenius algebra in $\mathcal{C}$. It turns out that one can extract such a Frobenius algebra from any solution C to Problem 6.6. In fact, one gets one such algebra for every element of $\mathcal{B}$. Let us have a brief look how this works (this amounts to what is said in section I:3.2, using the language of operator product expansions in CFT).

Let $\{\phi_{ij}, \mathcal{B}, \psi_{k,ab}\}$ and C be a solution to Problem 6.6. Pick your favourite boundary condition $a \in \mathcal{B}$. As an object in $\mathcal{C}$, the Frobenius algebra will be given by

$$A = \bigoplus_{k \in \mathcal{I}} U_k^{\oplus \dim_{\Bbbk} \psi_{k,aa}} \quad . \tag{6.8}$$

We can then identify $\psi_{k,aa} \cong \mathrm{Hom}(U_k, A)$. Next consider a world sheet X given by a disc with three boundary insertions $\Psi_i = (a, a, k_i, \psi_i, p_i, [\gamma_i])$, $i = 1, 2, 3$. To this, C assigns an element $C(\mathrm{X})$ in the space of states for an sphere with three marked points $(U_{k_1}, +)$, $(U_{k_2}, +)$ and $(U_{k_3}, +)$, which by construction of the 3dTFT is isomorphic to $\mathrm{Hom}(U_{k_1} \otimes U_{k_2} \otimes U_{k_3}, \mathbf{1})$. We determine an element $c \in \mathrm{Hom}(A \otimes A \otimes A, \mathbf{1})$ by requiring that $C(\mathrm{X}) = c \circ (\psi_1 \otimes \psi_2 \otimes \psi_3) \in \mathrm{Hom}(U_{k_1} \otimes U_{k_2} \otimes U_{k_3}, \mathbf{1})$. Repeating this construction for a disc with one and two insertions yields elements $\varepsilon \in \mathrm{Hom}(A, \mathbf{1})$ and $g \in \mathrm{Hom}(A \otimes A, \mathbf{1})$.

The morphism ε will be the counit of the Frobenius algebra A. Since g is non-degenerate by requirement (ii) on C, it provides an isomorphism $\phi^{-1} \in \mathrm{Hom}(A^\vee, A)$. This isomorphism can be used to construct morphisms $m \in \mathrm{Hom}(A \otimes A, A)$ and $\Delta \in \mathrm{Hom}(A, A \otimes A)$ from c, as well as $\eta \in \mathrm{Hom}(\mathbf{1}, A)$ from ε. For example, g and c are then related to m and ε via $g = \varepsilon \circ m$ and $c = \varepsilon \circ m \circ (m \otimes id_A)$.

One now has to verify that $(A, m, \eta, \Delta, \varepsilon)$ is a symmetric Frobenius algebra. Associativity, coassociativity and the Frobenius property (5.6) can be derived by applying condition (v) to cut a four-point function on the disc into two sets of three-point functions on the disc. The unit and counit property also follows from (v), this time by using it to cut a two-point function on the disc into a three-point and a one-point function.

Specialness of A is a little more tricky. Here one has to pass to a world sheet which has the topology of an annulus with one insertion of a boundary field, cf. the argument in I:3.2.

The outcome of this rather sketchy presentation is that each choice of $a \in \mathcal{B}$ allows us to extract a special symmetric Frobenius algebra A from C. Turning the argument around, it is a reasonable ansatz to start from such an algebra and try to construct a choice of field data as well as an assignment C which then gives a solution to Problem 6.6.

6.3 Choice of field data

In the remainder of this text we want to understand how a choice of field data and an assignment $\mathrm{X} \mapsto C(\mathrm{X})$ for *oriented* topological world sheets can be obtained from a symmetric special Frobenius algebra A. As starting point, we will define a choice of field data in terms of natural quantities obtained from A. To do so, we will need a little more notation (cf. section IV:2.2).

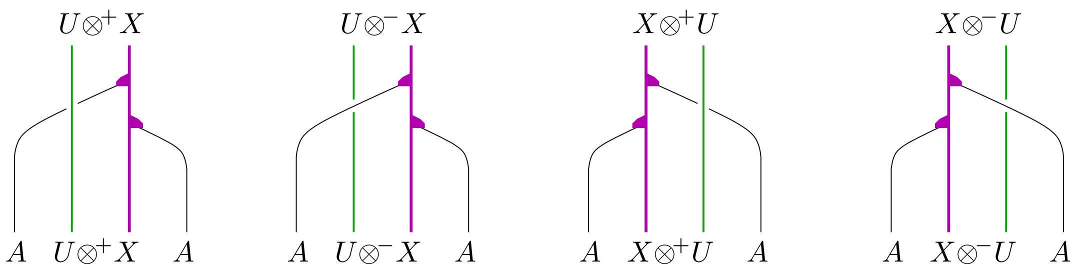
Definition 6.7:
Let $\mathcal{C}$ be a braided tensor category and A an algebra in $\mathcal{C}$. Given an object U of $\mathcal{C}$ and an A-bimodule $X = (\dot{X}, \rho_l, \rho_r)$, we define the A-bimodules $U \otimes^{\pm} X$ as

$$\begin{aligned} U \otimes^{+} X &:= (U \otimes \dot{X}, (id_U \otimes \rho_l) \circ (c_{U,A}^{-1} \otimes id_X), id_U \otimes \rho_r) \qquad \text{and} \\ U \otimes^{-} X &:= (U \otimes \dot{X}, (id_U \otimes \rho_l) \circ (c_{A,U} \otimes id_X), id_U \otimes \rho_r)\,, \end{aligned} \tag{6.9}$$

Similarly the bimodules $X \otimes^{\pm} U$ obtained by tensoring with U from the right are defined as

$$\begin{aligned} X \otimes^{+} U &:= (\dot{X} \otimes U, \rho_l \otimes id_U, (\rho_r \otimes id_U) \circ (id_X \otimes c_{U,A})) \qquad \text{and} \\ X \otimes^{-} U &:= (\dot{X} \otimes U, \rho_l \otimes id_U, (\rho_r \otimes id_U) \circ (id_X \otimes c_{A,U}^{-1}))\,. \end{aligned} \tag{6.10}$$

In graphical notation, the left/right action of A on $U \otimes^{\pm} X$ and $X \otimes^{\pm} U$ reads



(6.11)

The bimodules defined above are related to α-induced bimodules (see section 5.4) via $\alpha_A^{\pm}(V) = A \otimes^{\pm} V$.

If $\mathcal{C}$ is semisimple and A is special Frobenius, then $\mathcal{C}_A$ is semisimple [FuS, Proposition 5.24]. Also, for A special Frobenius, every A-module is submodule of an induced module [FuS, Lemma 4.15], so that if $\mathcal{C}$ only has finitely many isomorphism classes of simple objects, then so has $\mathcal{C}_A$. In our case $\mathcal{C}$ is modular, so that indeed $|\mathcal{I}| < \infty$. Let $\{\, M_\mu \,|\, \mu \in \mathcal{J} \,\}$ be a choice of representatives for the finite set of isomorphism classes of simple left A-modules.

Definition 6.8:
Let A be a symmetric special Frobenius algebra in a modular tensor category $\mathcal{C}$. The choice of field data $\mathcal{U}_A$ is given by taking the set of boundary conditions $\mathcal{B}(A)$ to be a set of representatives of isomorphism classes of A-modules, i.e. $\mathcal{B}(A) \cong \mathbb{Z}_{\geq 0}\mathcal{J}$. For $a \in \mathcal{B}$ we denote the corresponding A-module by M_a. Further,

$$\phi_{ij}(A) = \mathrm{Hom}_{A|A}(U_i \otimes^{+} A \otimes^{-} U_j, A) \quad \text{and} \quad \psi_{k,ab}(A) = \mathrm{Hom}_A(M_a \otimes U_k, M_b)\;. \tag{6.12}$$

This does certainly seem ad hoc, but the construction of the assignment $X \mapsto C(X)$ below will let the choice $\mathcal{U}_A$ appear in a more natural light. For the moment, just notice that as a special case of Lemma IV:2.2 we have

$$\mathrm{Hom}_{A|A}(A \otimes^- V, A \otimes^+ U^\vee) \cong \mathrm{Hom}_{A|A}(U \otimes^+ A \otimes^- V, A) \quad . \tag{6.13}$$

Combining this with (6.12) gives

$$Z_{ij}(A) := \dim_{\Bbbk} \phi_{ij}(A) = \dim_{\Bbbk} \mathrm{Hom}_{A|A}(A \otimes^- U_j, A \otimes^+ U_{\bar{\imath}}) = \tilde{Z}(A)_{\bar{\imath}j} \quad , \tag{6.14}$$

where $\tilde{Z}(A)_{kl}$ was defined in (5.31). As pointed out in remark 6.2, the matrix $Z_{ij}(A)$ should commute with the matrices $\hat{S}$ and $\hat{T}$. This is indeed the case, as follows from Theorem 5.24 (v) if one uses in addition that $Z(A) = \mathsf{C}\tilde{Z}(A)$, as well as the fact that $[\hat{S}, \mathsf{C}] = 0 = [\hat{T}, \mathsf{C}]$, where $\mathsf{C}_{kl} = \delta_{k,\bar{l}}$ is the charge conjugation matrix.

6.4 The assignment $X \mapsto C(X)$

In this section, A is again a symmetric special Frobenius algebra in a modular tensor category $\mathcal{C}$. Given an oriented topological world sheet X, we would like to construct an element $C_A(\mathrm{X}) \in \mathcal{H}(\hat{\mathrm{X}})$ such that the assignment $\mathrm{X} \mapsto C_A(\mathrm{X})$ gives a set of oriented correlators. The element $C_A(\mathrm{X})$ will be given through the action of the 3dTFT on a particular cobordism $\mathrm{M_X}$ with embedded ribbon graph $\mathrm{R_X}$, both to be defined below.

The connection manifold $\mathrm{M_X}$

Definition 6.9:
Given a topological world sheet X, the *connecting manifold* is defined as [FFFS]

$$\mathrm{M_X} = \hat{\mathrm{X}} \times [-1,1]/\sim \qquad \text{where} \quad ([x,\mathrm{or}],t) \sim ([x,-\mathrm{or}],-t) \quad . \tag{6.15}$$

This definition also applies to unoriented topological world sheets. If X is oriented, as in our case, the definition of $\mathrm{M_X}$ simplifies to

$$\mathrm{M_X} = \mathrm{X} \times [-1,1]/\sim \qquad \text{where} \quad (x,t) \sim (x,-t) \text{ for all } x \in \partial\mathrm{X} \quad . \tag{6.16}$$

Thus in words, for oriented world sheets, the connecting manifold is obtained by taking an interval $[-1,1]$ above each point of X and "folding" the interval back to itself over the boundary of X (see (IV:3.12) for an illustration).

In both, the oriented and unoriented case, there is a natural embedding $\iota_\mathrm{X} : \mathrm{X} \hookrightarrow \mathrm{M_X}$ given by

$$\iota_\mathrm{X}(x) = \big[[x,\pm\mathrm{or}],0\big] \quad , \tag{6.17}$$

and the boundary of $\mathrm{M_X}$ is just the double $\hat{\mathrm{X}}$ (as a manifold)

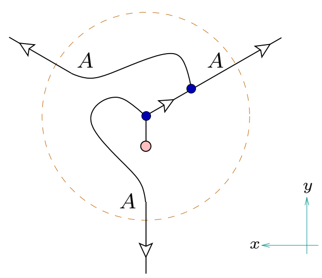
$$\partial\mathrm{M_X} = \big(\hat{\mathrm{X}}{\times}\{1\} \sqcup \hat{\mathrm{X}}{\times}\{-1\}\big)/\sim\, = \hat{\mathrm{X}} \quad . \tag{6.18}$$

Embedded ribbon graph $\mathrm{R_X}$

Next, we choose $\mathrm{M_X}$ to have weight zero and provide it with a ribbon graph R_X, turning it into a morphism $\mathrm{M_X} : \emptyset \to \hat{\mathrm{X}}$ of extended surfaces. The construction [6] of R_X involves a number of arbitrary choices, but the linear map $Z(\mathrm{M_X}) : \Bbbk \to \mathcal{H}(\hat{\mathrm{X}})$ will be shown to be independent of these choices. In steps (i)–(viii) below, we will always think of X as embedded in $\mathrm{M_X}$ via ι_X.

(i) Choose a triangulation T of X which has two- or three-valent vertices and faces with an arbitrary number of edges – Choice #1. The choice of T is subject to the following conditions.
- The boundary ∂X is covered by edges of T.
- Two-valent vertices in T are only allowed at marked points of X and every marked point on X coincides with a two-valent vertex of T.
- For a bulk insertion at p with arc germ $[\gamma]$, there has to be a representative γ of $[\gamma]$ such that γ is covered by edges of T (see e.g. figure (IV:4.26))

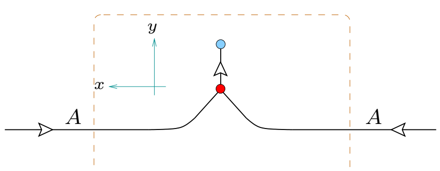
(ii) On each three-valent vertex in the interior of X place the following fragment of ribbon graph with three outgoing A-ribbons,

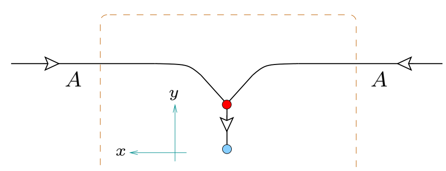


(6.19)

such that the orientation of X agrees with the one indicated in the figure. There are three possibilities to do this (rotating the graph) – Choice #2.

(iii) One each edge of T which does not lie on ∂X, place one of the following two fragments of ribbon graph with two ingoing A-ribbons such that the orientation of X agrees with the one indicated in the figure – Choice #3.





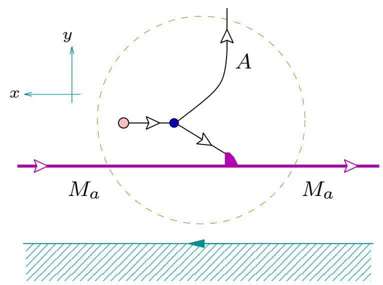
(6.20)

(iv) The edges on the boundary ∂X get labelled by elements of $\mathcal{B}$ as follows. If an edge e of T lies on a connected component of X without field insertion, it gets labelled by the element of $\mathcal{B}$ assigned to that boundary component (recall Definition 6.1 (ii)). Otherwise, e lies between two (not necessarily distinct) boundary insertions. In this case it gets labelled by $b = c$, using the convention in figure (6.1).

[6] When comparing to [I]– [IV], it has to be taken into account that in [I, II] a slightly different convention for orientations as in [IV] and the present text has been used. In short, in [I, II] the surface of the embedded ribbons carries the same orientation as the world sheet, while in [IV] and the present text, it carries the opposite orientation. More details on the relation between the two conventions are given in section IV:3.1.

(v) On each edge on the boundary ∂X place a ribbon labelled by M_a, with a the label assigned to that edge. The orientations of the ribbon core and surface have to be opposite to those of ∂X and X, respectively.

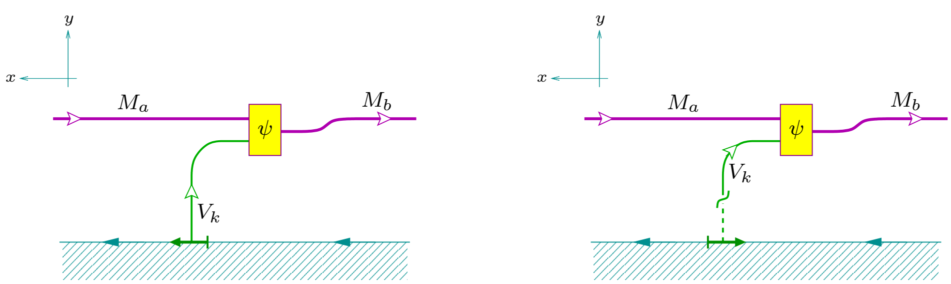
(vi) On each three-valent vertex on the boundary ∂X place ribbon graph fragment



(6.21)

such that bulk and boundary orientation indicated in the figure agree with those of X. Here a is the label of the two edges lying on the boundary ∂X, as assigned in (iv) (they have the same label by construction). Shown in (6.21) is a horizontal section of the connecting manifold as displayed in figure (IV:3.12). Correspondingly the lower boundary in (6.21) is that of $\mathrm{M_X}$ while the ribbons M_a are placed on the boundary of X as embedded in $\mathrm{M_X}$. The arrow on the boundary in (6.21) indicates the orientation of ∂X (transported to $\partial\mathrm{M_X}$ along the preferred intervals).

(vii) Let $v \in \partial$X be a two-valent vertex of T and let $\Psi = (a, b, k, \psi, p, [\gamma])$ be the corresponding boundary insertion. At v place one of the two ribbon graph fragments

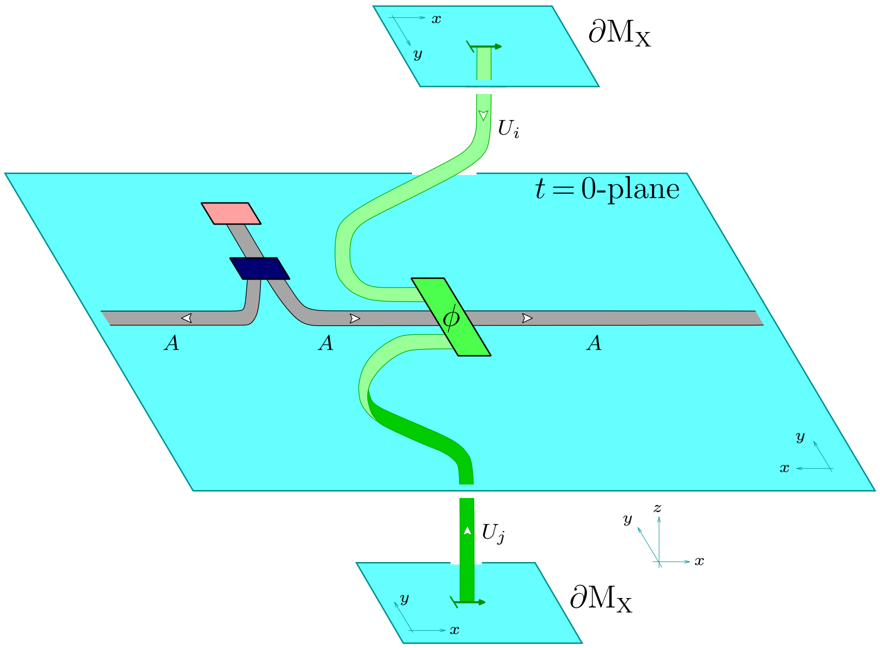


(6.22)

depending on the relative orientation of the arc-germ $[\gamma]$ and the boundary ∂X, and such that bulk and boundary orientation indicated in the figure agree with those of X. (The dashed-line notation refers to the "black" side of a ribbon, see figure (II:3.3)).

(viii) Let v be a two-valent vertex of T in the interior of X and let let $\Phi = (i, j, \phi, p, [\gamma], \mathrm{or}_2(p))$

be the corresponding bulk insertion. Place the following fragment of ribbon graph at v,



(6.23)

s.t. the 2-orientation in the t=0-plane and the orientations of the arcs on ∂M_X are as indicated in the figure. (Also, in this picture, the A-ribbons are viewed white side up).

Remark 6.10:
(i) This is the prescription for oriented world sheets. The modifications necessary in the unoriented case are described in II:3.1 and IV:3.

(ii) The prescription might seem complicated at first glance, but is in fact rather straightforward and probably best understood by looking at some examples, such as the topological world sheet being a torus (section I:5.3), an annulus (section I:5.8), and various correlators on the disc and on the sphere (sections IV:4.2–IV:4.4).

(iii) The above procedure is analogous to the definition of a two-dimensional lattice TFT in section 2.3. In fact, a 2d lattice TFT is obtained as a special case when choosing $\mathcal{C} = \mathcal{V}ect_f(\Bbbk)$.

Definition of C_A

As before, we denote by M_X also the morphism obtained by taking the connecting manifold with embedded ribbon graph R_X and weight zero. Then $Z(M_X)$ is a map $\Bbbk \to \mathcal{H}(\hat{X})$. We define, for an oriented topological word sheet X,

$$C_A(X) \;=\; Z(M_X)1 \quad \in \mathcal{H}(\hat{X}) \tag{6.24}$$

The ribbon graph R_X embedded in M_X, as constructed above, depends on the choices made in steps (i)–(iii). In order for the assignment (6.24) to be well-defined we need to verify that all choices lead to *equivalent* ribbon graphs in the sense that the invariant $Z(M_X)$ assigned to the morphism $M_X : \emptyset \to \hat{X}$ is independent of these choices.

This independence has been shown in several steps in sections I:5.1, IV:3.2 and IV:3.3 (for unoriented world sheets one needs in addition section II:3.1). Indeed, independence of the choices involved was the guiding principle in identifying the correct algebraic objects to be used in $\mathcal{U}_A$ and $C_A(\,\cdot\,)$. For example,

▪ the two elements in step (iii) are the natural ribbon graph fragments to insert at an edge of the triangulation. However, there is no preferred choice, and in order to guarantee independence of $C_A(\,\cdot\,)$ from choice #3, A is required to be symmetric, compare to figures (5.8) and (5.9).

▪ that all three rotations of the vertex in step (ii) lead to equivalent ribbon graphs follows from symmetry and coassociativity of A, see (I:5.9) for a figure with the required identity.

▪ any two triangulations can be related by a sequence of fusion and bubble moves, as displayed in (2.10). That two triangulations related by a single fusion move lead to equivalent ribbon graphs follows from the various associativity properties of A (i.e. associativity of m, coassociativity of Δ and the Frobenius property stating that Δ is an A-bimodule morphism). If two triangulations are related by a single bubble move, the corresponding ribbon graphs are equivalent due to A being special.

▪ if the fusion move involves an edge on the boundary, equivalence of the ribbon graphs is guaranteed by the representation property of the representation morphism ρ_{M_a} of the corresponding A-module M_a, see (I:5.12) for a figure.

▪ to relate any two triangulations in the presence of field insertions, we have to be able to move the three-valent vertices of the triangulation past the two-valent vertices whose position is fixed by the marked points. For a marked point on the boundary, this is possible because the morphism ψ inserted in (6.22) is an intertwiner of A-modules, see figure (IV:3.15). For a marked point in the bulk, the corresponding moves are possible, because the morphism ϕ in (6.23) is in $\mathrm{Hom}_{A|A}(U_i {\otimes^+} A {\otimes^-} U_j, A)$, with $\otimes^+$ and $\otimes^-$ in precisely this order (since an A-ribbon arriving from the "left" has to under the U_i-ribbon, while an A-ribbon arriving from the "right" as to stay above the U_j-ribbon, see (IV:3.25) for an illustration).

Solution to problem 6.6

We have following theorem; it was announced in [FuRS] and the details of its proof are presented in [FFRS].

Theorem 6.11:
Let A be a symmetric special Frobenius algebra in a modular tensor category $\mathcal{C}$. The choice of field data $\mathcal{U}_A$ (as in Definition 6.8) together with the assignment $\mathrm{X} \mapsto C_A(\mathrm{X})$ given in (6.24) is a set of oriented correlators.

Remark 6.12:
(i) An instructive example is the case where X is the torus without field insertions. It is treated in detail in section I:5.3.

(ii) Problem 6.4 can be solved in a similar fashion. In this case the required datum is a Jandl algebra (Definition 5.10 (iii)). The construction of the ribbon graph for unoriented world sheets without marked points is given in section II:3.1. Marked points are then treated in section IV:3. The statement and proof of the corresponding version of Theorem 6.11 can be found in [FFRS].

(iii) The converse statement "Every solution to Problem 6.6 is of the form $(\mathcal{U}_A, C_A)$" is an important open point. We have seen in section 6.2 that the presence of a Frobenius algebra is immediate. The difficulty lies is showing that this fixes the rest of the structure uniquely, in particular the correlator of three bulk fields on the sphere.

(iv) Let us call two sets of oriented correlators $(\mathcal{U}, C)$ and $(\mathcal{U}', C')$ *equivalent* if there are isomorphisms between the data $(\phi_{ij}, \mathcal{B}, \psi_{k,ab})$ and $(\phi'_{ij}, \mathcal{B}', \psi'_{k,ab})$ s.t. for all topological world sheets X we have $C(\mathrm{X}) = C'(\mathrm{X}')$, where X' is obtained from X by using the above isomorphism to relabel boundary conditions and field insertions.
Two non-isomorphic symmetric special Frobenius algebras A and A' can lead to equivalent sets of oriented correlators. In fact, it turns out that if A and A' are Morita-equivalent, then $(\mathcal{U}_A, C_A)$ and $(\mathcal{U}_{A'}, C_{A'})$ are equivalent.

6.5 Outlook

After presenting the construction of rational CFT correlators via 3dTFT and symmetric special Frobenius algebras in braided tensor categories, let us comment on some points in [I]–[IV] not discussed in this text and on directions for future studies.

A point not mentioned in this text was that apart from conformal boundary conditions, one can study also conformal defect lines. In the simplest case, these are marked circles on the world sheet. On the algebraic side they correspond to bimodules of the symmetric special Frobenius algebra A. We thus have a natural interpretation of A-modules (as boundary conditions) and A-bimodules (labelling defect lines). It turns out that bimodules are related to group-like and order-disorder symmetries of the CFT. For further details and references consult sections I:5.10, II:3.8, III:5 and IV:3.4, as well as [FFRS3].

In chapter 5 we have seen that the categories $\mathcal{C}(G, \psi, \Omega)$ are useful to construct examples. Given a modular tensor category, we can consider all invertible objects, i.e. objects U such that $U \otimes U^\vee \cong \mathbf{1}$ (cf. section 3.3). The subcategory generated by direct sums of such objects (called the *Picard category* in Definition III:2.1) is equivalent to one of the $\mathcal{C}(G, \psi, \Omega)$. Symmetric special Frobenius algebras in a Picard category can be analysed by group-cohomological methods, and this is the subject of [III]. In the conformal field theory literature, invertible objects are referred to as simple currents [SY].

Also, the relationship to an approach to euclidean rational CFT based on weak Hopf algebras [BPPZ, PZ1, PZ2] has not be touched upon in this text; regarding this point, the reader is referred to [FFRS4].

A pressing point for future studies is to make precise the relation between the complex-analytic part of the construction of a euclidean CFT and the algebraic part, which is the subject of [I]–[IV].

Another important aim is to develop tools to address the question of classification of modular tensor categories and of the Morita classes of symmetric special Frobenius algebras in them.

Using the fact that Davydov–Yetter cohomology of the pair $\mathcal{M}, \mathcal{C}$ can be expressed in terms of Hochschild cohomology of a certain Hopf algebra, it was shown in [ENO] that rational conformal field theories cannot be deformed within the class of rational conformal field theories.

Still, it would be highly interesting to study the deformations of CFTs. To this end it is at least necessary to generalise the approach presented in this text to compact, but not necessarily rational CFTs.

Altogether, the works [I]–[IV] and [C] represent a significant advance in the understanding of the structure of CFTs and of the corresponding questions arising in the study of algebras in braided tensor categories, and provide many relevant directions for further research.

References

[I] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators I: Partition functions*, Nucl. Phys. B 646 (2002) 353 [hep-th/0204148]

[II] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators II: Unoriented surfaces*, Nucl. Phys. B 678 (2004) 511 [hep-th/0306164]

[III] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators III: Simple currents*, Nucl. Phys. B 694 (2004) 277 [hep-th/0403158]

[IV] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators IV: Structure constants and correlation functions*, Nucl. Phys. B 715 (2005) 539 [hep-th/0412290]

[C] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, *Correspondences of ribbon categories*, Adv. Math. 199 (2006) 192 [math.CT/0309465]

[Ab1] L. Abrams, *Two-dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory and its Ramif. 5 (1996) 569

[Ab2] L. Abrams, *Modules, comodules and cotensor products over Frobenius algebras*, preprint math.RA/9806044

[At1] M.F. Atiyah, *Topological quantum field theories*, Publ. Math. I.H.E.S. 68 (1988) 175

[BP] C. Bachas and P.M.S. Petropoulos, *Topological models on the lattice and a remark on string theory cloning*, Commun. Math. Phys. 152 (1993) 191 [hep-th/9205031]

[BK] B. Bakalov and A.A. Kirillov, *Lectures on Tensor Categories and Modular Functors* (American Mathematical Society, Providence 2001)

[BPPZ] R.E. Behrend, P.A. Pearce, V.B. Petkova, and J.-B. Zuber, *Boundary conditions in rational conformal field theories*, Nucl. Phys. B 579 (2000) 707

[BPZ] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. B 241 (1984) 333

[BN1] Bion-Nadal, *Subfactor of the hyperfinite II_1 factor with Coxeter graph E_6 as invariant*, J. Operator Theory 28 (1992) 27

[BN2] Bion-Nadal, *An example of a subfactor of the hyperfinite* II_1 *factor whose pricipal grpah invariant is the Coxter graph E_6*, in Curent Topics in Operator Algebras (Nara, 1990), pp. 104, World Sci. Publishing, River Edge, NJ, 1991.

[BE1] J. Böckenhauer and D.E. Evans, *Modular invariants from subfactors: Type I coupling matrices and intermediate subfactors*, Commun. Math. Phys. 213 (2000) 267 [math.OA/9911239]

[BE2] J. Böckenhauer and D.E. Evans, *Modular invariants from subfactors*, Contemp. Math. 294 (2002) 95 [math.OA/0006114]

[BE3] J. Böckenhauer and D.E. Evans, *Modular invariants and subfactors*, Fields Institute Commun. 30 (2001) 11 [math.OA/0008056]

[BEK1] J. Böckenhauer, D.E. Evans, and Y. Kawahigashi, *On α-induction, chiral generators and modular invariants for subfactors*, Commun. Math. Phys. 208 (1999) 429 [math.OA/9904109]

[BEK2] J. Böckenhauer, D.E. Evans, and Y. Kawahigashi, *Chiral structure of modular invariants for subfactors*, Commun. Math. Phys. 210 (2000) 733 [math.OA/9907149]

[BEK3] J. Böckenhauer, D.E. Evans, and Y. Kawahigashi, *Longo–Rehren subfactors arising from α-induction*, Publ. RIMS 37 (2001) 1 [math.OA/0002154]

[B] R.E. Borcherds, *Vertex algebras, Kac–Moody algebras, and the monster*, Proc. Natl. Acad. Sci. USA 83 (1986) 3068

[BMT] D. Buchholz, G. Mack, and I.T. Todorov, *The current algebra on the circle as a germ of local field theories*, Nucl. Phys. B (Proc. Suppl.) 5B (1988) 20

[BFV] R. Brunetti, K. Fredenhagen, and R. Verch, *The generally covariant locality principle – A new paradigm for local quantum physics*, Commun. Math. Phys. 237 (2003) 31 [math-ph/0112041]

[CIZ] A. Cappelli, C. Itzykson, and J.-B. Zuber, *The A-D-E classification of minimal and $A_1^{(1)}$ conformal invariant theories*, Commun. Math. Phys. 113 (1987) 1

[C1] J.L. Cardy, *Conformal invariance and surface critical behavior*, Nucl. Phys. B 240 (1984) 514

[C2] J.L. Cardy, *Operator content of two-dimensional conformally invariant theories*, Nucl. Phys. B 270 (1986) 186

[C3] J. L. Cardy, *Conformal Invariance And Statistical Mechanics*, in: Les Houches Summer School (1988) 169.

[CR] C.W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebra s* (Wiley Interscience, New York 1962)

[DM] P. Deligne and J.S. Milne, *Tannakian categories*, Springer Lecture Notes in Mathematics 900 (1982) 101

[DMS] P. di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory* (Springer Verlag, New York 1996)

[D] R. Dijkgraaf, *A geometrical approach to two-dimensional conformal field theory*, Ph.D. thesis (Utrecht 1989)

[DLM] C. Dong, H. Li, and G. Mason, *Modular-invariancece of trace functions in orbifold theory and generalized moonshine*, Commun. Math. Phys. 214 (2000) 1 [q-alg/9703016]

[DHR] S. Doplicher, R. Haag, and J.E. Roberts, *Fields, observables and gauge transformations I*, Commun. Math. Phys. 13 (1969) 1

[DR1] S. Doplicher and J.E. Roberts, *Endomorphisms of C^*-algebras, cross products and duality for compact groups*, Ann. Math. 130 (1989) 75

[DR2] S. Doplicher and J.E. Roberts, *A new duality theory for compact groups*, Invent. math. 98 (1989) 157

[EM] S. Eilenberg and S. MacLane, *Cohomology theory of abelian groups and homotopy theory I–IV*, Proc. Natl. Acad. Sci. USA 36 (1950) 443, 657, 37 (1951) 307, 38 (1952) 325

[ENO] P.I. Etingof, D. Nikshych, and V. Ostrik, *On fusion categories*, preprint math.QA/0203060

[EP] D.E. Evans and P.R. Pinto, *Subfactor realisation of modular invariants*, Commun. Math. Phys. 237 (2003) 309 [math.OA/0309174]

[FFFS] G. Felder, J. Fröhlich, J. Fuchs, and C. Schweigert, *Correlation functions and boundary conditions in RCFT and three-dimensional topology*, Compos. Math. 131 (2002) 189 [hep-th/9912239]

[FFRS] J. Fjelstad, J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators V: Proof of modular invariance and factorization*, preprint to appear

[FhRS] K. Fredenhagen, K.-H. Rehren, and B. Schroer, *Superselection sectors with braid group statistics and exchange algebras, I: General theory*, Commun. Math. Phys. 125 (1989) 201

[FLM] I.B. Frenkel, J. Lepowsky, and A. Meurman, *Vertex Operator Algebras and the Monster* (Academic Press, New York 1988)

[FB] E. Frenkel and D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*, second edition (American Mathematical Society, Providence, in press)

[FrS] D.H. Friedan and S.H. Shenker, *The analytic geometry of two-dimensional conformal field theory*, Nucl. Phys. B 281 (1987) 509

[FG1] J. Fröhlich and F. Gabbiani, *Braid statistics in local quantum theory*, Rev. Math. Phys. 2 (1990) 251

[FG2] J. Fröhlich and F. Gabbiani, *Operator algebras and conformal field theory*, Commun. Math. Phys. 155 (1993) 569

[FFRS2] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, *Algebras in tensor categories and coset conformal field theories*, Fortschritte d. Phys. 52 (2004) 672 [hep-th/0309269]

[FFRS3] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, *Kramers–Wannier duality from conformal defects*, Phys. Rev. Lett. 93 (2004) 070601 [cond-mat/0404051]

[FFRS4] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, *Picard groups in rational conformal field theory*, preprint math.CT/0411507

[FK] J. Fröhlich and T. Kerler, *Quantum Groups, Quantum Categories and Quantum Field Theory* [Springer Lecture Notes in Mathematics 1542] (Springer Verlag, Berlin 1993)

[FK] J. Fröhlich and C. King, *The Chern–Simons theory and knot polynomials*, Commun. Math. Phys. 126 (1989) 167

[FuRS] J. Fuchs, I. Runkel, and C. Schweigert, *Conformal correlation functions, Frobenius algebras and triangulations*, Nucl. Phys. B 624 (2002) 452 [hep-th/0110133]

[FSS] J. Fuchs, A.N. Schellekens, and C. Schweigert, *Galois modular invariants of WZW models*, Nucl. Phys. B 437 (1995) 667 [hep-th/9410010]

[FuS] J. Fuchs and C. Schweigert, *Category theory for conformal boundary conditions*, Fields Institute Commun. 39 (2003) 25 [math.CT/0106050]

[FHK] M. Fukuma, S. Hosono, and H. Kawai, *Lattice topological field theory in two dimensions*, Commun. Math. Phys. 161 (1994) 157 [hep-th/9212154]

[GG] M.R. Gaberdiel and P. Goddard, *Axiomatic conformal field theory*, Commun. Math. Phys. 209 (2000) 549 [hep-th/9810019]

[Ga1] K. Gawędzki, Séminaire Bourbaki No.704, *Conformal field theory*, Astérisque 177-178 (1989) 95

[Ga2] K. Gawędzki, *Lectures on Conformal Field Theory*, in Quantum Fields and Strings: A Course For Mathematicians (P. Deligne, P. Etingof, D.S. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D.R. Morrison and E. Witten, eds.,), 2 vols., AMS (1999).

[Go] P. Goddard, *Meromorphic conformal field theory*, in: *Infinite-dimensional Lie Algebras and Groups*, V.G. Kac, ed. (World Scientific, Singapore 1989), p. 556

[Ha] R. Haag, *Local Quantum Physics* (Springer Verlag, Berlin 1996), 2nd ed.

[HK1] P. Hu and I. Kriz, *Conformal field theory and elliptic cohomology*, Adv. Math. 189 (2004) 325

[HK2] P. Hu and I. Kriz, *Closed and open conformal field theories and their anomalies*, Commun. Math. Phys. 254 (2005) 221 [hep-th/0401061]

[Hu] Y.-Z. Huang, *Two-dimensional Conformal Geometry and Vertex Operator Algebras* (Birkhäuser, Boston 1997)

[Hu1] Y.-Z. Huang, *Vertex operator algebras, the Verlinde conjecture and modular tensor categories*, preprint math.QA/0412261

[HL] Y.-Z. Huang and J. Lepowsky, *Tensor products of modules for a vertex operator algebra and vertex tensor categories*, in: *Lie Theory and Geometry*, R. Brylinski, J.-L. Brylinski, V. Guillemin, and V.G. Kac, eds. (Birkhäuser, Boston 1994), p. 349 [hep-th/9401119]

[I1] M. Izumi, *Subalgebras of infinite C^*-algebras with finite Watatani indices I. Cuntz algebras*, Commun. Math. Phys. 155 (1993) 157

[I2] M. Izumi, *On flatness of the Coxeter graph E_8*, Pacific J. Math. 166 (1994) 305

[JS1] A. Joyal and R. Street, *The geometry of tensor calculus, I*, Adv. Math. 88 (1991) 55

[JS2] A. Joyal and R. Street, *Braided tensor categories*, Adv. Math. 102 (1993) 20

[Kc] V.G. Kac, *Vertex Algebras for Beginners* (American Mathematical Society, Providence 1996)

[KS] L. Kadison and A.A. Stolin, *Separability and Hopf algebras*, Contemp. Math. 259 (2000) 279

[Ks] C. Kassel, *Quantum Groups* (Springer Verlag, New York 1995)

[KRT] C. Kassel, M. Rosso, and V. Turaev, *Quantum Groups and Knot Invariants* (Soc. Math. de France, Paris 1997)

[KLM] Y. Kawahigashi, R. Longo, and M. Müger, *Multi-interval subfactors and modularity of representations in conformal field theory*, Commun. Math. Phys. 219 (2001) 631 [math.OA/9903104]

[KO] A.A. Kirillov and V. Ostrik, *On q-analog of McKay correspondence and ADE classification of $\widehat{\mathfrak{sl}}(2)$ conformal field theories*, Adv. Math. 171 (2002) 183 [math.QA/0101219]

[Ko] J. Kock, *Frobenius algebras and 2D topological quantum field theories* (Cambridge University Press, Cambridge 2003)

[La] C.I. Lazaroiu, *On the structure of open-closed topological field theory in two dimensions*, Nucl. Phys. B 603 (2001) 497 [hep-th/0010269]

[LL] J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and their Representations* (Birkhäuser, Boston 2004)

[Lo] R. Longo, *A duality for Hopf algebras and for subfactors. I*, Commun. Math. Phys. 159 (1994) 133

[LRe1] R. Longo and K.-H. Rehren, *Nets of subfactors*, Rev. Math. Phys. 7 (1995) 567

[LRe2] R. Longo and K.-H. Rehren, *Local fields in boundary CFT*, Rev. Math. Phys. 16 (2004) 909 [math-ph/0405067]

[LRo] R. Longo and J.E. Roberts, *A theory of dimension*, K-Theory 11 (1997) 103 [funct-an/9604008]

[Ma] G. Mack, *Introduction to conformal invariant quantum field theory in two and more dimensions*, in: *Non-perturbative quantum field theory*, G. t'Hooft et al. (eds.) Proceedings of the Cargèse Summer School 1987 (Plenum Press 1988).

[ML] S. Mac Lane, *Categories for the Working Mathematician* (Springer Verlag, New York 1971)

[Mi] P.D. Mitchener, *C^*-categories*, Proceedings of the London Mathematical Society 84 (2002) 375.

[Mo] G. Moore, *Some comments on branes, G-flux, and K-theory*, Int. J. Mod. Phys. A 16 (2001) 936 [hep-th/0012007]

[MS1] G. Moore and N. Seiberg, *Polynomial equations for rational conformal field theories*, Phys. Lett. B 212 (1988) 451

[MS2] G. Moore and N. Seiberg, *Naturality in conformal field theory*, Nucl. Phys. B 313 (1989) 16

[MS3] G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, Commun. Math. Phys. 123 (1989) 177

[Mu1] M. Müger, *Conformal field theory and Doplicher–Roberts reconstruction*, Fields Institute Commun. 30 (2001) 297 [math-ph/0008027]

[Mu2] M. Müger, *From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories*, J. Pure Appl. Alg. 180 (2003) 81 [math.CT/0111204]

[NTV] D. Nikshych, V. Turaev, and L. Vainerman, *Quantum groupoids and invariants of knots and 3-manifolds*, Topology and its Appl. 127 (2003) 91

[Oc] A. Ocneanu, *Quantized groups, string algebras and Galois theory for algebras*, in: *Operator Algebras and Applications*, Vol. 2 [London Math. Soc. Lecture Note Series # 136], D.E. Evans and M. Takesaki, eds. (Cambridge University Press, Cambridge 1988), p. 119

[Os] V. Ostrik, *Module categories, weak Hopf algebras and modular invariants*, Transform. Groups 8 (2003) 177 [math.QA/0111139]

[Pa1] B. Pareigis, *Non-additive ring and module theory I. General theory of monoids*, Publ. Math. Debrecen 24 (1977) 189

[Pa2] B. Pareigis, *Non-additive ring and module theory II. C-categories, C-functors, and C-morphisms*, Publ. Math. Debrecen 24 (1977) 351

[Pa3] B. Pareigis, *Non-additive ring and module theory IV. The Brauer group of a symmetric monoidal category*, Springer Lecture Notes in Mathematics 549 (1976) 112

[Pa4] B. Pareigis, *Non-additive ring and module theory III. Morita theorems*, Publ. Math. Debrecen 25 (1978) 177

[Pa5] B. Pareigis, *Morita equivalence of module categories with tensor products*, Commun. in Algebra 9 (1981) 1455

[Pa6] B. Pareigis, *On braiding and dyslexia*, J. Algebra 171 (1995) 413

[PZ1] V.B. Petkova and J.-B. Zuber, *The many faces of Ocneanu cells*, Nucl. Phys. B 603 (2001) 449 [hep-th/0101151]

[PZ2] V.B. Petkova and J.-B. Zuber, *Conformal field theories, graphs and quantum algebras*, in: *MathPhys Odyssey 2001 – Integrable Models and Beyond*, M. Kashiwara and T. Miwa, eds. (Birkhäuser, Boston 2002), p. 415 [hep-th/0108236]

[Pi] R.S. Pierce, *Associative Algebras* (Springer Verlag, New York 1980)

[Pl] J. Polchinski, *String Theory, vol.I* (Cambridge University Press, Cambridge 1998)

[Py] A.M. Polyakov, *Conformal symmetry of critical fluctuations*, Sov. Phys. JETP Lett. 12 (1970) 538

[Pp] Popa, S., *Classification of subfactors and their endomorphisms*, CBMS Regional Conference Series in Mathematics, **86**, AMS, Providence, RI, 1995.

[Q] F. Quinn, *Lectures on axiomatic topological quantum field theory*, in: *Geometry and Quantum Field Theory*, D.S. Freed and K.K. Uhlenbeck, eds. (American Mathematical Society, Providence 1995), p. 323

[Re] K.-H. Rehren, *Locality and modular invariance in 2D conformal QFT*, Fields Institute Commun. 30 (2001) 341 [math-ph/0009004]

[RT1] N.Yu. Reshetikhin and V.G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Commun. Math. Phys. 127 (1990) 1

[RT2] N.Yu. Reshetikhin and V.G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. math. 103 (1991) 547

[Sa] N. Saavedra Rivano, *Catégories Tannakiennes* [Springer Lecture Notes in Mathematics 265] (Springer Verlag, New York 1972)

[SY] A.N. Schellekens and S. Yankielowicz, *Simple currents, modular invariants, and fixed points*, Int. J. Mod. Phys. A 5 (1990) 2903

[Se1] G.B. Segal, *Two-dimensional conformal field theories and modular functors*, in: *IXth International Congress on Mathematical Physics*, B. Simon, A. Truman, and I.M. Davis, eds. (Adam Hilger, Bristol 1989), p. 22

[Se2] G.B. Segal, *The definition of a conformal field theory*, in: *Topology, Geometry and Quantum Field Theory: Proceedings of the 2002 Oxford Symposium in Honour of the 60th Birthday of Graeme Segal*, U.L. Tillmann, ed. (Cambridge Univ. Press 2004), p. 421.

[SW] R.F. Streater and A.S. Wightman, *PCT, Spin and Statistics, and All That* (Addison–Wesley, Reading 1989)

[Th] W. Thirring, *A soluble relativistic field theory*, Ann. Phys. 3 (1958) 91

[Tu] V.G. Turaev, *Quantum Invariants of Knots and 3-Manifolds* (de Gruyter, New York 1994)

[Tu1] V.G. Turaev, *Modular categories and 3-manifold invariants*, Int. J. Mod. Phys. B 6 (1992) 1807

[VZ] F. Van Oystaeyen and Y.H. Zhang, *128*, J. Algebra 202 (1998) 96 The Brauer group of a braided monoidal category

[Va] C. Vafa, *Conformal theories and punctured surfaces*, Phys. Lett. B 199 (1987) 195

[Ve] E. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, Nucl. Phys. B 300 (1988) 360

[Vo] A.A. Voronov, *Topological field theories, string backgrounds and homotopy algebras*, in: Proceedings of the *XXIIth International Conference on Differential Geometric Methods in Theoretical Physics*, J. Keller et al., eds. (Universidad Nacional Autonoma de Mexico, Mexico 1994), p. 167 [hep-th/9401023]

[Wa] A. Wassermann, *Operator algebras and conformal field theory III. Fusion of positive energy representations of $LSU(N)$ using bounded operators*, Invent. math. 133 (1998) 467

[Wl] K.G. Wilson, *Non-Lagrangian models of current algebra*, Phys. Rev. 179 (1969) 1499

[Wt1] E. Witten, *Topological quantum field theory*, Commun. Math. Phys. 117 (1988) 353

[Wt2] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys. 121 (1989) 351

[Wt3] E. Witten, *On holomorphic factorization of WZW and coset models*, Commun. Math. Phys. 144 (1992) 189

[X] F. Xu, *New braided endomorphisms from conformal inclusions*, Commun. Math. Phys. 192 (1998) 349

[Z] Y. Zhu, *Modular invariance of characters of vertex operator algebras*, J. Amer. Math. Soc. 9 (1996) 237