Hints and solutions for problem sheet # 11

Advanced Algebra — Winter term 2016/17

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Problem 45

$1 \Rightarrow 3$ By Theorem 2.3.2, it remains to show that f^* is surjective. Let $\varphi : L \to J$ be given. Since f is injective, by definition of injective modules, there is $\tilde{\varphi} : M \to J$ such that $\tilde{\varphi} \circ f = \varphi$, that is, $f^*(\tilde{\varphi}) = \varphi$.

$3 \Rightarrow 2$ Let $J \to M \to N$ be a short exact sequence. By 3, applied to this sequence, we have that $f^* : \mathrm{Hom}_R(M, J) \to \mathrm{Hom}_R(J, J)$ is surjective. Hence there is $\varphi : M \to J$ such that $f^*(\varphi) = id_J$. But this means that $\varphi \circ f = id_J$, i.e. φ is a splitting map for the sequence $J \to M \to N$.

Problem 46

1. If it would split, then $\mathbb{Z}/p^2\mathbb{Z} = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, which is not true because the first has an element of (additive) order p^2, while the latter do not. Thus by Theorem 5.1.2, $\mathbb{Z}/p\mathbb{Z}$ is not projective. On the other hand, the submodule $\langle p \rangle$ of $\mathbb{Z}/p^2\mathbb{Z}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.
2. See Cor 5.1.5 – since $\mathbb{Z}$ is a PID, projective implies free. We have already shown $\mathbb{Q}$ to not be free over $\mathbb{Z}$, so it cannot be projective over $\mathbb{Z}$.
3. "Sometimes": If the ring is semisimple (e.g. if it is a field), by problem 47, every module, finitely-generated or not, is already injective.

   For the ring $\mathbb{Z}$, injective modules are divisible groups. A non-zero divisible abelian group A is never finitely generated. Indeed, a finitely generated abelian group is isomorphic to $\mathbb{Z}^r \times \prod_{i=1}^n \mathbb{Z}/m_i\mathbb{Z}$ for some m_i, which is not divisible.

Problem 47

3 implies 1,2,4 (and the condition in the extra problem):
By Thm. 4.4.1 (3), every short exact sequence splits (why?). By Thm. 5.1.2 (2) and Thm. 5.2.1 (2), every module is both projective and injective. In particular every simple one.

1 implies 3:
By Thm. 5.1.2 (2), every short exact sequence splits. Thus every submodule is a direct summand, and by Thm. 4.4.1, every module is semisimple.

2 implies 3:
Same as above, using Thm. 5.2.1 (2).

4 implies 3:
It is enough to show that R is semisimple (Prop. 4.4.8). Let $\{T_i\}_{i\in I}$ be all simple submodules of R. Let $S := \sum_{i\in I} T_i$. If $S = R$, by Thm. 4.4.1 M is semisimple. Suppose not. Pick a maximal left ideal I of R containing S (exists by Cor. 2.4.8). Then R/I is simple, hence projective, and $I \to R \to R/I$ splits. Thus $R = I \oplus X$ with X simple. But the span of all simple modules is contained in I. Contradiction.

Extra problem: There are rings for which every simple module is injective, but which are not semisimple. Here is an example:

Let X be an infinite set and let $R = \mathrm{Fun}(X, \mathbb{F}_2)$, where $\mathbb{F}_2$ is the field with two elements. Then R is a commutative ring, and for all $x \in R$, $x \cdot x = x$ (such a ring is called *boolean*). Furthermore, as $R = \prod_{p\in X} \mathbb{F}_2$, we know from Sheet 9, Problem 39 that it is not semisimple.
Let M be a maximal ideal in R (exists by Lem. 2.4.7). Then $S := R/M$ is a simple R-module, and every simple R-module is isomorphic to one of these (Lem. 4.2.1).

Claim: S is injective.
Proof: By Lem. 5.2.2 it suffices to show that for every ideal $I \subset R$, every R-module homomorphism $f : I \to S$ extends to an R-module homomorphism $\tilde{f} : R \to S$.
Let thus $I \subset R$ and $f : I \to S$ be given. Note that for every $m \in M$ and $x \in I$ we have $0 = m.f(x) = f(mx)$, that is, $MI \subset \ker(f)$.
Suppose $I \subset M$. Then $I = II \subset MI$ (the equality follows from idempotency of all elements of R). Thus $I \subset \ker(f)$, i.e. $f = 0$. Hence in this case we can choose $\tilde{f} = 0$.
Suppose I is not a subset of M. Since M is maximal, we must have $M + I = R$. Note that $MI = M \cap I$, since certainly $MI \subset M \cap I$, and any $x \in M \cap I$ can be written as $x = x \cdot x$, so that also $M \cap I \subset MI$.
Consider the map $g : M \oplus I \to S$, $g(m, i) = f(i)$. The kernel of g is $\ker(g) = M \oplus \ker(f) \supset M \oplus MI$. On the other hand, the kernel of $\pi : M \oplus I \to M + I$, $(m, i) \mapsto m + i$ is $\ker(\pi) = \{(x, -x) | x \in M \cap I\}$. Let $\iota : M \cap I \to M \oplus I$, $x \mapsto (x, -x)$ such that $\ker(\pi) = \mathrm{im}(\iota)$. Since $g \circ \iota(x) = g(x, -x) = f(-x) = 0$ for $x \in M \cap I = MI$, we have $\ker \pi \subset \ker g$. Thus g factors through $\bar{g} : M + I \to S$:

$$
\begin{tikzcd}
M \cap I \arrow[r, "\iota"] & M \oplus I \arrow[r, "\pi"] \arrow[d, "g"'] & M + I \arrow[dl, dashed, "\exists!\tilde{g}"] \\
& S &
\end{tikzcd}
$$

By construction, the restriction of $\tilde{g}$ to I is equal to f.

Problem 48

Let Z be an R-Module with maps $\gamma : Z \to A, \delta : Z \to B$ such that $f \circ \gamma = g \circ \delta$. In particular, that means that $\mathrm{im}(\gamma \times \delta) \subseteq M'$. This inclusion is the map to M factoring the maps γ and δ.

Uniqueness: remember that $\alpha : M' \to A, \beta : M' \to B$ are just projection onto the two coordinates. So, any map $\chi : Z \to M'$ making things commute will have to satisfy $\alpha\chi(z) = \delta(z)$ and $\beta\chi(z) = \gamma(z)$, i.e., be of the form $z \mapsto (\delta(z), \gamma(z))$ (i.e. is uniquely as we specified).

Problem 49

1. Let $b_3 \in B_3$.

$$\begin{array}{lll} \Rightarrow & \exists a_4 \in A_4 \text{ with } t_4(a_4) = g_3(b_3) & (t_4 \text{ surjective}) \\ \Rightarrow & g_4 g_3(b_3) = 0 = g_4 t_4(a_4) = t_5 f_4(a_4) & \text{(exactness and commutativity of diagram)} \\ \Rightarrow & f_4(a_4) = 0 & (t_5 \text{ injective}) \\ \Rightarrow & \exists a_3 \in A_3 \text{ with } f_3(a_3) = a_4 & \text{(exactness of top row)} \\ \Rightarrow & g_3(b_3 - t_3(a_3)) = g_3(b_3) - g_3 t_3(a_3) = & \\ & = t_4(a_4) - t_4 f_3(a_3) = t_4(a_4) - t_4(a_4) = 0 & \text{(= 's from above)} \\ \Rightarrow & \exists b_2 \in B_2 \text{ with } g_2(b_2) = b_3 - t_3(a_3) & \text{(exactness of bottom row)} \\ \Rightarrow & \exists a_2 \in A_2 \text{ with } t_2(a_2) = b_2 & (t_2 \text{ surjective}) \end{array}$$

Then

$$t_3(f_2(a_2) + a_3) = t_3 f_2(a_2) + t_3(a_3) = g_2 t_2(a_2) + t_3(a_3) = g_2(b_2) + t_3(a_3) = b_3 - t_3(a_3) + t_3(a_3) = b_3$$

2. Let $a_3 \in A_3$ with $t_3(a_3) = 0$.

$$\begin{array}{lll} \Rightarrow & t_4 f_3(a_3) = g_3 t_3(a_3) = 0 & \text{(commutativity of diagram)} \\ \Rightarrow & f_3(a_3) = 0 & (t_4 \text{ injective}) \\ \Rightarrow & \exists a_2 \in A_2 \text{ with } f_2(a_2) = a_3 & \text{(exactness of top row)} \\ \Rightarrow & g_2 t_2(a_2) = t_3 f_2(a_2) = t_3(a_3) = 0 & \text{(commutativity of diagram)} \\ \Rightarrow & \exists b_1 \in B_1 \text{ with } g_1(b_1) = t_2(a_2) & \text{(exactness of bottom row)} \\ \Rightarrow & \exists a_1 \in A_1 \text{ with } t_1(a_1) = b_1 & (t_1 \text{surjective}) \\ \Rightarrow & g_1 t_1(a_1) = g_1(b_1) = t_2 f_1(a_1) = t_2(a_2) & \text{(commutativity of diagram)} \\ \Rightarrow & f_1(a_1) = a_2 & (t_2 \text{injective}) \\ \Rightarrow & f_2(a_2) = f_2 f_1(a_1) = 0 = a_3 & \end{array}$$