**Hints and solutions for problem sheet # 10**
**Advanced Algebra — Winter term 2016/17**
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**Problem 40**

Recall the explicit definition of the tensor product $M \otimes_R N$ in Proposition 4.5.1. It is given by $B/\langle S_R \rangle$ where

$$B = \bigoplus_{(m,n) \in M \times N} \mathbb{Z}\,(m,n)$$

and

$$\begin{aligned} S_R = \big\{ & (m+m',n) - (m,n) - (m',n) \ , \\ & (m,n+n') - (m,n) - (m,n') \ , \\ & (m.r,n) - (m,r.n) \qquad \big| \, m,m' \in M \ , \ n,n' \in N \ , \ r \in R \big\} \ . \end{aligned}$$

Note that $B$ does not depend on $R$. The claim follows if we can show that $S_R = S_P$. We have

$$\begin{aligned} S_P = \big\{ & (m+m',n) - (m,n) - (m',n) \ , \\ & (m,n+n') - (m,n) - (m,n') \ , \\ & (m.\pi(p),n) - (m,\pi(p).n) \qquad \big| \, m,m' \in M \ , \ n,n' \in N \ , \ p \in P \big\} \ . \end{aligned}$$

Since $\pi(p) \in R$, clearly $S_P \subset S_R$. Since the map $\pi : P \to R$ is surjective, each element $(m.r,n) - (m,r.n)$ also occurs in $S_P$. Thus $S_P = S_R$.

**Problem 41**

1. We use the universal property of $\otimes_R$: Defined $g : M \times N \to M \otimes_S N$, $(m,n) \mapsto (r.m) \otimes_S n$. This map is clearly bilinear. It furthermore satisfies, for all $s \in S$,

   $$g(m.s,n) = r.(m.s) \otimes_S n = (r.m).s \otimes_S n = r.m \otimes_S s.n = g(m,s.n) \ ,$$

   where we used that left and right action of a bimodule commute, and that $\otimes_S$ is balanced. Thus $f$ is balanced, and there exists a unique $\tilde{g} : M \otimes_S N \to M \otimes_S N$ such that $\tilde{g}(m \otimes_S n) = (r.m) \otimes_S n$. Take $\rho(r) = \tilde{g}$.

2. By linearity, it is enough to verify this on the set of generators $\{m \otimes_S n | m \in M, n \in N\}$ of $M \otimes_S N$. We have

   $$\begin{aligned} \rho(r)(\rho(r')(m \otimes_S n)) &= \rho(r)((r'.m) \otimes_S n) = (r.(r'.m)) \otimes_S n \\ &\overset{(*)}{=} ((rr').m)) \otimes_S n = \rho(rr')(m \otimes_S n) \ , \end{aligned}$$

   where we used associativity of the action in (*).

3. Abbreviate $h := f \otimes_S id$. $\mathbb{Z}$-linearity of $h$ is clear. We need to check that for all $r \in R$, $m \in M$, $n \in N$, $h(r.(m \otimes_S n)) = r.h(m \otimes_S n)$. By definition, $r.(m \otimes_S n) = \rho(r)(m \otimes_S n) = (r.m) \otimes_S n$. Thus

$$\begin{aligned} h(r.(m \otimes_S n)) &= h((r.m) \otimes_S n) = f(r.m) \otimes_S n = (r.f(m)) \otimes_S n \\ &= r.(f(m) \otimes_S n) = r.h(m \otimes_S n) \end{aligned}$$

**Problem 42**

1. *Example 1:* Take the complex number $\mathcal{C}$ with right action by multiplication, but with left action $x.y := x\bar{y}$, i.e. multiplying after complex conjugation.

   *Example 2:* Take $K[X]$ with left action multiplication and right action $p.q := p \cdot q(0)$, i.e. act with the constant term of the polynomial $q$.

2. From Prop. 4.5.7, $\iota(M) \otimes_R \iota(N)$ has a (unique) R-R bimodule structure such that $r \cdot (m \otimes n) = (r \cdot m) \otimes n$ and $(m \otimes n) \star t = m \otimes (n \star t)$. We need to check that $(m \otimes n) \star t = t.(m \otimes n)$:

$$\begin{array}{rcll} (m \otimes n) \star t & = & m \otimes (n \star t) & \text{Prop 4.5.7, tensoring R-R bimodules} \\ & = & m \otimes (t \cdot n) & \text{by definition of the bimodule action on } i({}_R N) \\ & = & (m \star t) \otimes n & \text{because } \otimes_R \text{ is balanced} \\ & = & (t.m) \otimes n & \text{by definition of the bimodule action on } i({}_R M) \\ & = & t \cdot (m \otimes n) & \text{by definition of the left action} \end{array}$$

**Problem 43**

1. Write $\beta(m, n) := m \otimes_R n$. Then $\beta$ is $R$-balanced:

$$\begin{array}{lrcl} (1) & \beta(m + m', n) & = & \beta(m, n) + \beta(m', n) \\ (2) & \beta(m, n + n') & = & \beta(m, n) + \beta(m, n') \\ (3) & \beta(r \cdot m, n) & = & \beta(m, r \cdot n) \end{array}$$

   To get from here to $R$-bilinear, it is enough to check $\beta(r \cdot m, n) = r \cdot \beta(m, n)$ (which implies $\beta(m, r \cdot n) = r \cdot \beta(m, n)$ because of (3)).

   But by definition of the $R$-module structure on $M \otimes_R N$, and of the $R$-$R$-bimodule structure on $M$, we have $r.(m \otimes n) = (r.m) \otimes n = (m.r) \otimes n$. Thus $r.\beta(m, n) = \beta(m.r, n)$.

2. $R$-bilinear implies $R$-balanced, so minimally have the maps as $\mathbb{Z}$-modules. That is, we have a unique $\tilde{\beta} : M \otimes_R N \to L$ which is a $\mathbb{Z}$-mod hom. factoring $\beta$. On simple tensors,

$$\tilde{\beta}((r.m \otimes n) = \beta(r.m, n) = r.\beta(m, n) = r.\tilde{\beta}(m \otimes n)$$

   where the middle equality holds because $\beta$ is $R$-linear (here, using only the property in the first variable). $\tilde{\beta}$ a $\mathbb{Z}$-mod hom implies in particular that it's additive and we can extend this result on simple tensors to all tensors, which establishes that $\tilde{\beta}$ is an $R$-mod hom.

**Problem 44**

1. Let $\pi : M \to M/IM$ be the canonical projection. Let $h := id \otimes_R \pi : R/I \otimes_R M \to R/I \otimes_R M/IM$. Consider the identity map $id : R/I \otimes_R M \to R/I \otimes_R M$. Since $R/I \otimes_R IM$ is zero in $R/I \otimes_R M$ (why?), the identity map factors through the map $g : R/I \otimes_R M/IM \to R/I \otimes_R M$. It is easy to check on elements that $g$ and $h$ are inverse to each other.

   Thus $R/I \otimes_R M \cong R/I \otimes_R M/IM$. Now use Lemma 4.5.4 to see that $R/I \otimes_R M/IM = R/I \otimes_{R/I} M/IM$. The latter $R$-module in turn is isomorphic to $M/IM$. This shows the claim.

2. We will show that $R/(I+J)$ satisfies the universal property of the tensor product.

   Define $t : R/I \times R/J \to R/(I+J)$ by $t(r+I, s+J) = rs + (I+J)$. (Show that) $t$ is an $R$-balanced map. Let $A$ be an abelian group with an $R$-balanced map $f : R/I \times R/J \to A$. Define $g : R/(I+J) \to A$ via $g(r + (I+J)) = f(r+I, 1+J)$ (why is $g$ well-defined?). Then $g$ is an abelian-group-hom, and since $f$ is $R$-balanced, have

$$\begin{array}{rclcl} gt(r+I, s+J) & = & g(r \cdot s + (I+J)) & = & f(r \cdot s + I, 1+J) \\ = f((r+I)s, 1+J) & = & r(r+I, s \cdot (1+J)) & = & f(r+I, s+J) \end{array}$$

   Thus $gt = f$ and $g$ is uniquely determined since $t$ maps $R/I \times R/J$ onto $R(I+J)$. Thus $R/(I+J)$ and $t$ are a tensor product of $R/I$ and $R/J$.

   Uniqueness of tensor products give us the isomorphism of abelian groups in the claim. We can also see write the isomorphism explicitly as

$$r + I + J \mapsto (r+I) \otimes (1+J) \ .$$

3. Apply part 2 to get that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m\mathbb{Z} + n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(m, n)$.