Hints and solutions for problem sheet # 09

Advanced Algebra — Winter term 2016/17

(Ingo Runkel)

Problem 35

1 implies 2 is clear.
2 implies 1: need to show: the intersection of M_i with sum of M_j over all $j \neq i$ is 0. Suppose not. Then $m_i = \sum_j m_j$ for some m_i in M_i and m_j in M_j. But the sum has to be finite. Hence there is already a finite subset for which the sum is not direct.

Problem 36

Submodule L: This follows from 3 of Theorem 4.4.1. Think of L as a submodule of M. Then $M = L \oplus X$ for X also a submodule of M. All submodules of L are also submodules of M. Take a submodule $S \subset L$. This gives rise to a decomposition of M, $M = S \oplus Y$. The induced decomposition of L then follows by intersection: $L = L \cap M = L \cap (S \oplus Y) = S \oplus (L \cap Y)$. Then S satisfies 3 and is thus semisimple.
Quotient module N: We may take $N = M/L$. L a submodule, by 4.4.1 (3), can write $M = L \oplus X$ for some X also a submodule. That means that $N = M/L \cong (L \oplus X)/L \cong X$, so is a submodule of M and therefore semisimple by the first part.
Converse statement: This is false in general.
Example 1: Take $\mathbb{C}[X]$. In Problem 34 we saw that $\mathbb{C}[X]$ possesses indecomposable modules M of length 2. Such a module has a simple submodule $\mathbb{C}_\lambda$ and a simple quotient, which is again $\mathbb{C}_\lambda$, for some $\lambda \in \mathbb{C}$, i.e. it sits in a short exact sequence $\mathbb{C}_\lambda \to M \to \mathbb{C}_\lambda$.
Example 2: Consider the short exact sequence $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z}$ of $\mathbb{Z}$-modules. Here, $\mathbb{Z}/2\mathbb{Z}$ is simple as a $\mathbb{Z}$-module, but $\mathbb{Z}/4\mathbb{Z}$ is not.

Problem 37

Note that $P|_U = \mathrm{Id}$ and $P(V) = U$.
Claim 1: Let $x \in G$. We need to show that $\bar{P}(xv) = x\bar{P}(v)$. Let, for $g \in G$ $h := x^{-1}g$. Then

$$\bar{P}(xv) = \frac{1}{|G|}\sum_{g\in G}(gPg^{-1})(xv) = \frac{1}{|G|}\sum_{h\in G} x(hPh^{-1})(v) = x\bar{P}(v).$$

Claim 2: For $u \in U$ and $g \in G$, we have $gu \in U$, therefore $P(gu) = gu$. Now

$$\bar{P}(u) = \frac{1}{|G|}\sum(gPg^{-1})u = \frac{1}{|G|}(gP(g^{-1}u)) = \frac{1}{|G|}\sum gg^{-1}u = \frac{1}{|G|}\sum u = u$$

Let $u \in V$. Then $\bar{P}(u) \in U$ and it follows that $\bar{P}^2(v) = \bar{P}(v)$.

Now let $W' := \ker(\bar{P})$. Then, as $\bar{P}$ is a $K[G]$-hom. W' is a $K[G]$-module. By idempotence of $\bar{P}$, V decomposes as

$$V = \ker(\bar{P}) \oplus \ker(\bar{P} - 1) = W' \oplus U.$$

Problem 38

1. The centre of $R \times S$ is $Z(R) \times Z(S)$. Composing the embedding $\iota : K \to Z(R) \times Z(S)$ with the projections $R \times S \to R$ and $R \times S \to S$ (which are ring-homomorphisms) gives the K-algebra structures on R and S.

2. Suppose $M \in Z(\mathrm{Mat}_n(R))$. Then $E_{ij} M E_{kl} = M E_{ij} E_{kl} = \delta_{j,k} M E_{il}$. But also $E_{ij} M E_{kl} = M_{jk} E_{il}$. Taking $i = l$ and summing over i gives $\delta_{j,k} M = M_{jk} I_{n \times n}$. Thus M is a multiple of the identity matrix. A matrix of the form $r I_{n \times n}$ commutes with all matrices of the form $s I_{n \times n}$ ($r, s \in R$) iff $r \in Z(R)$.

   Let $\iota : K \to Z(\mathrm{Mat}_n(R))$ give the algebra structure on matrices. Then $\iota(k) = \varphi(k) I_{n \times n}$ for a unique $\varphi(k) \in Z(R)$. One checks that $\varphi : K \to Z(R)$ is a ring homomorphism. This gives the algebra structure on R.

3. *Claim:* Let A be a semisimple K-algebra. Then there are division algebras $D_1, \ldots, D_n$ over K and $m_1, \ldots, m_n > 0$ such that

   $$A \cong \mathrm{Mat}_{m_1}(D_1) \times \cdots \times \mathrm{Mat}_{m_n}(D_n)$$

   as K-algebras. The pairs (D_i, m_i) are unique up to ordering.

   Proof: The statement is true for the underlying rings. Since the lhs is a K-algebra, the ring-isomorphism induces a K-algebra structure on the rhs. By parts 1 and 2, the $D_1, \ldots, D_n$ are also K-algebras, hence division algebras over K. By construction, the ring isomorphism we started from is now an isomorphism of K-algebras.

   Uniqueness works in the same way as for rings in Proposition 4.4.13.

4. We know from Section 1.2 that all finite-dimensional division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. There are of dimension 1,2,4, respectively, over $\mathbb{R}$. The corresponding matrix altebras have dimension $n^2, 2n^2, 4n^2$ over $\mathbb{R}$. Hence we

must find all ways to write 9 as a sum of the numbers $1, 2, 4, 8, 9$. We have

$$\begin{aligned}
9 &= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \\
&= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 2 \\
&= 1 + 1 + 1 + 1 + 1 + 2 + 2 \\
&= 1 + 1 + 1 + 2 + 2 + 2 \\
&= 1 + 2 + 2 + 2 + 2 \\
&= 1 + 1 + 1 + 1 + 1 + 4 \\
&= 1 + 1 + 1 + 2 + 4 \\
&= 1 + 2 + 2 + 4 \\
&= 1 + 4 + 4 \\
&= 1 + 8 \\
&= 9
\end{aligned}$$

To each number except 4 there is a unique choice of matrix algebra in the corresponding direct sum decomposition. For 4 one has the choice between $\mathrm{Mat}_2(\mathbb{R})$ and $\mathbb{H}$. There are 7 sums without 4s, 3 sums with one 4 and 1 sum with two 4s. Thus there are $7 + 6 + 3 = 16$ isomorphism classes of 9-dimensional semisimple $\mathbb{R}$-algebras.

Problem 39

The infinite product S is never semisimple.

Solution 1: Since all R_i are non-zero, by Prop. 4.2.2 it has a simple module M_i. Then M_i is also an S-module: for $s := (r_j)_{j\in I}$ and $m \in M_i$ define $s.m := r_i.m$. Clearly, 1) M_i is simple also as an S-module, and 2) for $i \neq j$ we have $M_i \ncong M_j$ as S-modules (since R_i acts non-zero on M_i but zero on M_j).
Thus there is an infinite number of mutually non-isomorphic simple modules for S. By Prop. 4.4.8, S cannot be semisimple.

Solution 2: Consider $I = \bigoplus_{i\in I} R_i \subset S$. This is an S-submodule of ${}_S S$ (why?). Suppose there is a submodule J such that $I \oplus J = S$. Then $J \neq \{0\}$ and we can take a non-zero element $x \in J$. Since J is a submodule (i.e. a left ideal) of S, also $rx \in J$ for any choice of r. Let x_j be a non-zero entry of x. Take r to be the family which is 1 in entry j and zero else. Then rx is the family which is x_j in entry j and zero else. In particular, $rx \neq 0$ and $rx \in I$. Contradiction (to $I \cap J = \{0\}$).

Problem without points

1. For each integer $m \in \mathbb{Z}$ define $\rho_m : \mathbb{C}[G] \to \mathbb{C}$, $\rho_m([x]) = e^{2\pi i m x}$. Note that that does not depend on the representative $x \in \mathbb{R}$ of $[x] \in \mathbb{R}/\mathbb{Z}$.

2. Take $I = \ker(\rho_0)$. Explicitly, I consists of all elements of the form $\sum_{i=1}^n \lambda_i[x_i]$ such that $\sum_{i=1}^n \lambda_i = 0$. Suppose there is a complement J. Then J must be non-zero. Let $y \in J$, $y \neq 0$. We can write $y = \sum_{i=1}^n \mu_i[x_i]$ for some n, μ_i, x_i. Pick a real number r such that $[x_i + r] \neq [x_j]$ for all i, j. Since J is a submodule, also $z = \sum_{i=1}^n \mu_i[x_i + r]$ lies in J. But then also $y - z \in J$, and $y - z \neq 0$. But the sum of coefficients is 0, hence $y - z \in I$. Contradiction (to $I \cap J = \{0\}$).

3. No, because in this statement what one means by a representation is a *continuous* group homomorphism $G \to \mathrm{GL}(V)$ for some complex (or real) vector space V, and not about all representations.