**Hints and solutions for problem sheet # 07**
**Advanced Algebra — Winter term 2016/17**
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## Problem 26

1. Let $a \in \mathbb{Z}$ be non-zero. A non-stabilising descending chain of submodules is $M_m := a^m\mathbb{Z} \subset \mathbb{Q}$ for $m \in \mathbb{N}$. A non-stabilising ascending chain of submodules is $N_m := a^{-m}\mathbb{Z} \subset \mathbb{Q}$ for $m \in \mathbb{N}$.

2. *Claim:* Every proper submodule $M \subsetneq Q_p$ is of the form $p^{-n}\mathbb{Z}$ for some $n$.

   *Proof:* Suppose $m/p^k \in M$, where we assume that $m \neq 0$ and $m$ and $p^k$ have no common divisors (that is, $p$ does not divide $m$). Then there are $a, b \in \mathbb{Z}$ such that $am + bp^k = 1$, i.e. $a\frac{m}{p^k} + b = \frac{1}{p^k}$. Thus also $p^{-k} \in M$, and hence $p^{-k}\mathbb{Z} \subset M$. But if $M$ contains all $p^{-k}$, then $M = Q_p$, and otherwise there is a maximal such $k$ and $M = p^{-k}\mathbb{Z}$.

   Suppose there is a descending chain $M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$ in $Q_p$. We may suppose that all $M_i$ are different from $Q_p$. Then there are integers $n_i \geq 0$ with $M_i = p^{-n_i}\mathbb{Z}$ and $n_0 > n_1 > n_2 \ldots$. Clearly, there cannot be an infinite such chain, contradiction.

   The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is not artinian (and anyway not noetherian): Note that $Q_p \subset \mathbb{Q}/\mathbb{Z}$ is a $\mathbb{Z}$-submodule. Let $p_1 < p_2 < p_3 < p_4 < \ldots$ be a sequence of increasing prime numbers. Set

   $$X_n = \mathrm{span}\{Q_{p_k} | k \geq n\} \ .$$

   Then clearly $X_1 \supset X_2 \supset X_3 \supset \cdots$. In fact, each of these inclusions is proper: Let $x \in X_n$. Then

   $$x = \sum_{k=n}^{N} \frac{a_k}{p_k^{b_k}} \ ,$$

   for some large enough $N$, $a_k \in \mathbb{Z}$, $b_k \geq 0$. In particular, the denominator of $x$ can at most contain the prime factors $p_n, \cdots, p_N$, but it does not contain any of the prime factors $p_1, \ldots, p_{n-1}$.

## Problem 27

1. $\mathbb{Q} \subset \mathbb{R}$ works for both: $\mathbb{R}_{\mathbb{Q}}$ is an infinite-dimensional $\mathbb{Q}$-vector space and hence is neither noetherian or artinian. On the other hand ${}_{\mathbb{Q}}\mathbb{Q}$ and ${}_{\mathbb{R}}\mathbb{R}$ are both one-dimensional vector spaces (over $\mathbb{Q}$ and $\mathbb{R}$, respectively).

2. $R$ contains the two-sided ideal $I = \{\left(\begin{smallmatrix} 0 & s \\ 0 & 0 \end{smallmatrix}\right) | s \in S\}$. Consider the short exact sequence of (left or right) $R$-modules $I \to R \to R/I$. One checks that $R/I \cong S \oplus T$ with left or right action of $r = \left(\begin{smallmatrix} s & s' \\ 0 & t \end{smallmatrix}\right)$ on $(a,b) \in S \oplus T$ given by $r.(a,b) = (sa, tb) = (a,b).r$ (Details?).

   Below [prop] stands for "noetherian", or for "artinian".

   *R is left [prop]:* By Proposition 4.1.1, if $I$ and $S \oplus T$ are [prop], so is $R$.

   By assumption, ${}_S S$ and ${}_T T$ are [prop]. Therefore, also the left $R$-module $S \oplus T$ is [prop] (why?).

   The left $R$-action on $I$ is, for $r = \left(\begin{smallmatrix} s & s' \\ 0 & t \end{smallmatrix}\right)$ and $a \in S$,

$$r. \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & sa \\ 0 & 0 \end{pmatrix} .$$

   Since ${}_S S$ is [prop], so is ${}_R S$, and hence $I$.

   *R is not right [prop]:* By Proposition 4.1.1, if $I$ is not right [prop], neither is $R$. The right $R$-action on $I$ is, for $r = \left(\begin{smallmatrix} s & s' \\ 0 & t \end{smallmatrix}\right)$ and $a \in S$,

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} .r = \begin{pmatrix} 0 & at \\ 0 & 0 \end{pmatrix} .$$

   Since $S_T$ is not [prop], neither is $I$ (why?).

**Problem 28**

1. Clearly $X_1 \subset X_2 \subset X_3 \subset \dots$. Let $y \in \ker(f)$ be given. We will show $y = 0$.

   Set $z_1 = y$. Since $y \in N$ and $f(y) = 0$, $z_1 \in X_1$. Since $f$ is surjective, there is $z_2 \in N$ such that $z_1 = f(z_2)$. But then $f(f(z_2)) = f(z_1) = 0$, and so $z_2 \in X_2$. In this way one constructs $z_n \in X_n$ with $f^{n-1}(z_n) = y$ and $f^n(z_n) = 0$. Since $N$ is noetherian, there is $K$ such that $X_k = X_K$ for all $k \geq K$. In particular $z_{K+1} \in X_{K+1}$ is also an element of $X_K$. From $z_{K+1} \in X_{K+1}$ we get $f^K(z_{K+1}) = y$. From $z_{K+1} \in X_K$ we get $f^K(z_{K+1}) = 0$. Thus $y = 0$.

2. No. For example, take the $\mathbb{Z}$-module $Q_p$ from Problem 26. Set $R = \mathbb{Z}$, $N = M = Q_p$. The identity map $N \to M$ serves as the injection. Multiplication by $p$ is a surjection $Q_p \to Q_p$. But it is not injective, as e.g. $\frac{1}{p}$ gets mapped to zero.

**Problem 29**

1. Clearly, $\dim_K K_\lambda \leq 1$. On the other hand, the map $K_\lambda \to K$, $X \mapsto \lambda$ is well-defined and surjective. So $\dim_K K_\lambda \geq 1$.

   The $K[X]$-module structure on $K_\lambda$ is as follows: $K$ acts via scalar multiplication on the underlying $K$-vector space $K$, and $X$ acts by multiplication by $\lambda$.

Let $f : K_\lambda \to K_\mu$ be a $K[X]$-module map. Let $v$ a nonzero element of $K_\lambda$. Then $f$ must commute with the multiplication by $X$.

$$f(X \cdot v) = X \cdot f(v)$$

But in $K_\lambda$, we have $X \cdot v = \lambda v$. Whereas in $K_\mu$, $X \cdot f(v) = \mu f(v)$. Since $\lambda, \mu \in K$ and $K[X]$-mod hom is still a $K$-mod hom, we see that we'd need $f(X \cdot v) = f(\lambda v) = \lambda f(v)$ to be equal to $X \cdot f(v) = \mu f(v)$. As $K_\lambda, K_\mu$ are 1-dimensional, $f = 0$ unless $\lambda = \mu$.

2. Let $\varphi : K[X_1, \ldots, X_n] \to K$ be the evaluation homomorphism sending $X_i$ to $\lambda_i$. Its kernel is an ideal. Since $\varphi$ is surjective and $K$ is a field, this ideal is maximal. It remains to show that

$$(*) \qquad \ker \varphi = \langle X_1 - \lambda_1, \ldots, X_n - \lambda_n \rangle \ .$$

Since $\varphi(X_i - \lambda_i) = 0$, it is clear that $\ker \varphi \supset \langle X_1 - \lambda_1, \ldots, X_n - \lambda_n \rangle$. Equality is less obvious, and we use the following argument:

Consider the ring homomorphism $\psi : K[X_1, \ldots, X_n] \to K[Y_1, \ldots, Y_n]$, which is the identity on $K$ and sends $X_i$ to $Y_i + \lambda_i$. Note that $\psi$ is actually an isomorphism (why?). Then (*) is equivalent to $\psi(\ker \varphi) = \psi(\langle X_1 - \lambda_1, \ldots, X_n - \lambda_n \rangle)$. We have $\psi(\ker \varphi) = \ker(\varphi \circ \psi^{-1})$ (why?) and $\varphi' := \varphi \circ \psi^{-1}$ is the ring homomorhism $\varphi' : K[Y_1, \ldots, Y_n] \to K$ which is the identity on $K$ and sends all $Y_i$ to 0. Thus (*) is equivalent to

$$\ker \varphi' = \langle Y_1, \ldots, Y_n \rangle \ ,$$

which is obvious.