Hints and solutions for problem sheet # 06
Advanced Algebra — Winter term 2016/17
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Problem 21

Given $z \in Z(R)$ define the family $\alpha^r_M : M \to M$, $\alpha^r_M(m) = r.m$. We need to verify that for all R-modules M, N and all $f \in \mathrm{Hom}_R(M,N)$,

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle \alpha^r_M} & & \downarrow{\scriptstyle \alpha^r_N} \\ M & \xrightarrow{f} & N \end{array}$$

commutes. Indeed, $\alpha^r_N(f(m)) = r.f(m) = f(r.m) = f(\alpha^r_M(m))$. Thus α^r is a natural transformation and we obtain a map

$$\alpha : Z(A) \to \mathrm{End}(Id) \quad , \quad r \mapsto \alpha^r \ .$$

We will show that this map is injective and surjective.
Injective: Suppose $\alpha^r = \alpha^s$. Then also $\alpha^r_R(1) = \alpha^s_R(1)$. But $\alpha^r_R(1) = r.1 = r$, etc., and so $r = s$.
Surjective: Let $\eta \in \mathrm{End}(Id)$ be given. Set $r = \eta_R(1)$. We will show that $r \in Z(R)$. Indeed, for $s \in R$ arbitrary, consider the R-module homomorphism $g : R \to R$, $g(x) = xs$. Since η is natural, the square

$$\begin{array}{ccc} R & \xrightarrow{g} & R \\ \downarrow{\scriptstyle \eta_R} & & \downarrow{\scriptstyle \eta_R} \\ R & \xrightarrow{g} & R \end{array}$$

commutes. Evaluating on $1 \in R$ shows $\eta_R(g(1)) = g(\eta_R(1))$, i.e. $rs = sr$.
We now claim that $\eta = \alpha^r$. Let M be an R-module. We need to check that for all $m \in M$, $\eta_M(m) = r.m$. Consider the map $g_m : R \to M$, $r \mapsto r.m$. This is an R-module homomorphism (why?). As η is natural, the square

$$\begin{array}{ccc} R & \xrightarrow{g_m} & M \\ \downarrow{\scriptstyle \eta_R} & & \downarrow{\scriptstyle \eta_M} \\ R & \xrightarrow{g_m} & M \end{array}$$

commutes. Evaluating on 1 gives $\eta_M(g_m(1)) = g_m(\eta_R(1))$, i.e. $\eta_M(m) = r.m$.

Problem 22

Consider finite-dimensional vector spaces only and let $\alpha_U : U \to U^*$ be a collection of isomorphisms. Let $f : V \to W$ be a linear map. The only commuting diagram we can write down is

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\alpha_V"'] & W \arrow[d, "\alpha_W"] \\ V^* & W^* \arrow[l, "f^*"'] \end{tikzcd}$$

Clearly, there is no collection of isomorphism α_U which makes this commute for all f, just take $f = 0$.

Problem 23

We first define functors $F : \mathcal{N} \to k\text{-Mod}_{\text{fin}}$ and $G : k\text{-Mod}_{\text{fin}} \to \mathcal{N}$.
F is easy, just take $F(m) = K^m$ and for a matrix M let $F(M)$ be the linear map it represents. (Why is this a functor?)
G is more awkward. For each finite-dimensional vector space V choose a basis. We do this by fixing a linear isomorphism $\alpha_V : K^{\dim(V)} \to V$ for each V (why is this the same as choosing a basis?). The choice of α_V is arbitrary, except for that we insist that K^m gets its standard basis, i.e. $\alpha_{K^m} = id_{K^m}$ (where is this used below?). On objects, $G(V) = \dim(V)$. For a morphism $f : V \to W$ let $G(f) = \alpha_W^{-1} \circ f \circ \alpha_V : K^{\dim(V)} \to K^{\dim(W)}$, or rather the matrix representation of this map. Now one should write a few lines to check this is indeed a functor. (Details?)
Next we need to check that FG and GF are equivalent to the identity functor. Since for $m \in \mathbb{Z}_{\geq 0}$, $GF(m) = m$, and also for each matrix M, $GF(M) = M$ (why?), we have $GF = Id_{\mathcal{N}}$ and there is nothing more to do.
For the other composition, note that $FG(V) = K^{\dim(V)}$. Now we claim that the family $\alpha_V : K^{\dim(V)} \to V$ from above defines a natural isomorphism $\alpha : FG \to Id$. The relevant diagram is, for $f : V \to W$,

$$\begin{tikzcd} K^{\dim(V)} \arrow[r, "FG(f)"] \arrow[d, "\alpha_V"'] & K^{\dim(W)} \arrow[d, "\alpha_W"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

But by definition $FG(f) = \alpha_W^{-1} \circ f \circ \alpha_V$ (why?), so the diagram indeed commutes.

Problem 24

1. (**injective:**) Consider γ, γ' with the same image under $\psi_{M,A}$. That means that $\gamma(m)(1) = \gamma'(m)(1)$ (for all $m \in M$). Since γ is an R-module homomorphis, we know that $\gamma(r.m)(s) = \gamma(m)(sr)$ for all $m \in M$, $r, s \in R$. Thus, for all m, r,

$$\gamma'(m)(r) = \gamma'(r.m)(1) = \gamma(r.m)(1) = \gamma(m)(r) \ .$$

(**surjective:**) Consider $f \in \mathrm{Hom}_{\mathbb{Z}}(M, A)$. We will send this to $m \mapsto (r \mapsto f(r \cdot m)$ (why is this an RMod homomorphism?). Under $\psi_{M,A}$, this is sent to $m \mapsto f(1 \cdot m) = f(m)$, so we recover f.

2. For $\alpha : A \to A',\ \mu : M \to M'$

$$\begin{tikzcd}
\mathrm{Hom}_R(M', \mathrm{Hom}_{\mathbb{Z}}(R, A)) \arrow[r, "\gamma\mapsto(\mathrm{m}\mapsto(r\mapsto\alpha\circ\gamma(\mu(\mathrm{m}))(r))"] \arrow[d, "\psi_{M',A}"'] & \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, A')) \arrow[d, "\psi_{M,A'}"] \\
\mathrm{Hom}_{\mathbb{Z}}(M', A) \arrow[r, "f\mapsto\alpha\circ f\circ\mu"] & \mathrm{Hom}_{\mathbb{Z}}(M, A)
\end{tikzcd}$$

Element-wise, things are sent

$$\begin{array}{ccc}
\gamma & \mapsto & (\mathrm{m} \mapsto (r \mapsto \alpha \circ \gamma(\mu(\mathrm{m}))(r)) \\
\downarrow & & \downarrow \\
\mathrm{m}' \mapsto \gamma(\mathrm{m}')(1) & \mapsto & \mathrm{m} \mapsto \alpha \circ \gamma(\mu(\mathrm{m}))(1)
\end{array}$$

and it commutes.

Problem 25

We will guess the inverse and verify that it does the job. Try

$$F' \circ Id \xrightarrow{F'\eta} F'GF \xrightarrow{\varepsilon' F} Id \circ F \ .$$

It is enough to check on order of composition, the other follows by exchanging primed and unprimed quantities.

$$\begin{aligned}
&\big[F \circ Id \xrightarrow{F\eta'} FGF' \xrightarrow{\varepsilon F'} Id \circ F' \xrightarrow{=} F' \circ Id \xrightarrow{F'\eta} F'GF \xrightarrow{\varepsilon' F} Id \circ F\big] \\
&= \big[F \circ Id \xrightarrow{F\eta} FGF \xrightarrow{=} F \circ Id \circ GF \xrightarrow{F\eta' GF} FGF'GF \xrightarrow{FG\varepsilon' F} FG \circ Id \circ F \\
&\qquad \xrightarrow{=} FGF \xrightarrow{\varepsilon F} Id \circ F\big] \\
&= \big[F \circ Id \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} Id \circ F\big] \ = \ Id_F \ .
\end{aligned}$$

For the first equality, we repeatedly use identities like

$$\begin{aligned}
&\big[FGF' \xrightarrow{\varepsilon F'} Id \circ F' \xrightarrow{=} F' \circ Id \xrightarrow{F'\eta} F'GF\big] \\
&= \big[FGF' \xrightarrow{=} FGF' \circ Id \xrightarrow{FGF'\eta} FGF'GF \xrightarrow{\varepsilon F'GF} Id \circ F'GF \xrightarrow{=} F'GF\big] \ .
\end{aligned}$$

(Why does that hold?)
For the second and third equality one uses the defining properties of unit and counit.