Hints and solutions for problem sheet # 05

Advanced Algebra — Winter term 2016/17

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Problem 17

1. R-Mod has kernels and co-kernels, so purely as maps of k-vector spaces, we know that every $g : A \to B$ will have a kernel and cokernel. Now to produce filtrations of these.

   For the kernel: As a k-linear map, g has a kernel the sub-vector space of A given by $a \in A$ which are sent to 0 under g; $(\ker(g), K : \ker(g) \hookrightarrow A)$. As a result, the induced associated filtration of the kernel is defined as $\ker(g)_i := \ker(g) \cap A_i$.

   Proof that this is the kernel: Let $\alpha : U \to A$ be a $\mathbb{Z}$-filtered linear map such that $g \circ \alpha = 0$. We need to show that there is a unique $\mathbb{Z}$-filtered linear map $\tilde{\alpha} : U \to \ker(g)$ such that $K \circ \tilde{\alpha} = \alpha$. But this is clear from the definitions as in this case $\tilde{\alpha} = \alpha$, understood as a filtered linear map $U \to \ker(g)$.

   For the cokernel: Recall that as an k-vector space, the cokernel is of the form $(B/(\mathrm{im}(g), c : B \to B/(\mathrm{im}(g))$. Then the filtration on B_i induces one on the cokernel of g, via c; $(coker(g))_i := c(B_i)$.

   Proof that this is the cokernel: (is similar to kernel).

2. As on the underlying k-vector space k, f is the identity map, and mono and epi are immediate.

   All filtered linear maps $W \to V$ are zero, since to preserve filtration, they have to map $W_0 = k$ into $V_0 = 0$. The category of $\mathbb{Z}$-filtered vector spaces does not contain an inverse to f.

Problem 18

1. Write $R := \mathcal{C}(A, A)$. Since $\mathcal{C}$ is an Ab-category, R is an abelian group. The composition of $\mathcal{C}$ defines an associative composition $\circ : R \times R \to R$. By additivity of $\mathcal{C}$, the composition is bilinear, i.e. the distributive law holds. The identity $1_A \in \mathcal{C}(A, A)$ is the unit of R. (A one-object category can be additive only if its unique object is the zero object, so that $R = \{0\}$.)

2. Let $F : \mathcal{C} \to \mathbf{Ab}$ be an additive functor. Thus $F(\bullet) =: M$ is an abelian group, and $F : \mathcal{C}(\bullet, \bullet) \to \mathrm{End}(M)$ is a homomorphism of abelian groups. Since F preserves unit and composition, we have $F(1_R) = id_M$ and $F(f \circ g) = F(f) \circ F(g)$. Thus $F : R \to \mathrm{End}(M)$ is a ring homomorphism. By Problem 3, this amounts to turning M into an R-module.

   Conversely, given an R-module M we can set $F(\bullet) = M$ and use the group homomorphism $R \to \mathrm{End}(M)$ from Problem 3 to define F on morphisms.

3. Let $F, G : \mathcal{C} \to \mathbf{Ab}$ be the two additive functors. Write $M = F(\bullet)$ and $N = G(\bullet)$ for the corresponding R-modules. The natural transformation consists of a single map $\eta_\bullet =: f : M \to N$ (a group homomorphism, as it is a morphism in $\mathbf{Ab}$). The naturality square reads, for $r \in R = \mathcal{C}(\bullet, \bullet)$,

$$\begin{tikzcd} M \arrow[r, "F(r)"] \arrow[d, "f"'] & M \arrow[d, "f"] \\ N \arrow[r, "G(r)"] & N \end{tikzcd}$$

This commutes for all r if and only if f is an R-module homomorphism.

Problem 19

1. The morphism sets are

$$(\mathcal{C} \times \mathcal{D})((C_1, D_1), (C_2, D_2)) \quad := \quad \{(f, g) \in \mathcal{C}(C_1, C_2) \times \mathcal{D}(D_1, D_2)\}$$

Composition is coordinate-wise. Associativity and identity properties follow from those of $\mathcal{C}$ and $\mathcal{D}$.

2. Let F be a functor from the product-category $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$. Then $F_C : \mathcal{C} \to \mathcal{E}$ is given by, for $D, D' \in \mathcal{D}$ and $f : D \to D'$:

$$F_C(D) = F((C, D)) \quad , \quad F_C(f) = F((1_C, f)) \ .$$

The functor $F_D : \mathcal{D} \to \mathcal{E}$ is defined analogously. From functoriality of F one immediatly obtains that F_C, F_D are functors. For the commuting square note

$$\begin{aligned} F_{C'}(g) \circ F_D(f) &= F((1_{C'}, g)) \circ F((f, 1_D)) = F((1_{C'}, g) \circ (f, 1_D)) \\ &= F((f, g)) = F((f, 1_{D'})) \circ F((1_C, g)) = F_{D'}(f) \circ F_C(g) \ . \end{aligned}$$

3. We define F on objects $(C, D) \in \mathcal{C} \times \mathcal{D}$ as

$$F((C, D)) := F_D(C) = F_C(D) \ .$$

On morphisms $(f, g) : (C, D) \to (C', D')$ we define

$$F((f, g)) := F_{D'}(f) \circ F_C(g) = F_{C'}(g) \circ F_D(f) \ ,$$

where the equality is the commuting diagram we assume.

Now we need to show that F is a functor. We have $F((1_C, 1_D)) = F_D(1_C) \circ F_C(1_D) = 1_{(C,D)} \circ 1_{(C,D)} = 1_{(C,D)}$. For morphisms $(f, g) : (C, D) \to (C', D')$

and $(h,k) : (C', D') \to (C'', D'')$ we have

$$\begin{aligned} F((h,k) \circ (f,g)) &= F((h \circ f, k \circ g)) \\ &= F_{D''}(h \circ f) \circ F_C(k \circ g) \\ &= F_{D''}(h) \circ F_{D''}(f) \circ F_C(k) \circ F_C(g) \\ &\overset{(*)}{=} F_{D''}(h) \circ F_{C'}(k) \circ F_{D'}(f) \circ F_C(g) \\ &= F((h,k)) \circ F((f,g)) \ , \end{aligned}$$

where in (*) the commuting square was used.

Problem 20

1. Simpler than 2, so let's just look at 2:
2. For $f : A \to A'$, $x \in \mathcal{C}(A', B)$ the definition of $f^* : \mathcal{C}(A', B) \to \mathcal{C}(A, B)$ was $f^*(x) = x \circ f$. $\mathcal{C}(-, B)$ being a contravariant functor amounts to checking $id^*(x) = x$ and, for $g : A'' \to A$, $(f \circ g)^*(x) = g^*(f^*(x))$. The first identity is immediate, and for the second note that both sides are equal to $x \circ f \circ g$.
3. Given $f : A \to A'$ and $h : B \to B'$, and any $g \in \mathcal{C}(A, B)$, then $\mathcal{C}(id_A, h) : \mathcal{C}(A, B) \to \mathcal{C}(A, B')$ via $g \mapsto h \circ g$. $\mathcal{C}(f, id_B) : \mathcal{C}(A', B) \to \mathcal{C}(A, B)$ via $g \mapsto g \circ f$. And so on, so that we map $g \in \mathcal{C}(A, B)$ to $\mathcal{C}(A', B')$ via $g \mapsto h \circ g \mapsto (h \circ g) \circ f$ in one direction and $g \mapsto g \circ f \mapsto h \circ (g \circ f)$ in the other. Associativity of composition of morphisms means these two are equivalent, and we have the square as desired.