Hints and solutions for problem sheet # 03
Advanced Algebra — Winter term 2016/17
(Ingo Runkel)

Problem 8 (Misc)

1. $\mathrm{End}_R({}_RR) \cong R^{\mathrm{op}}$: The two ring homomorphisms are, for $f \in \mathrm{End}_R({}_RR)$, $r \in R$, $f \mapsto f(1)$ and $r \mapsto (-) \cdot r$ (details?).
   $\mathrm{End}_{R\text{-}R}({}_RR_R) \cong Z(R)$: The maps are the same as above. To see that $f(1) \in Z(R)$ note that $rf(1) = f(r1) = f(r) = f(1r) = f(1)r$, using that f is a module homomorphism for the left and right action.

2. That IM is a submodule is clear: with $im \in IM$ also $rim \in IM$, as $ri \in I$ for all $r \in R$.

   Let now $i \in I$ and $m + IM \in M/IM$. Then, by definition of the action on M/IM, $\rho(i)(m + IM) = i.m + IM$, which is 0 in M/IM.

3. ($\Leftarrow$) Immediate from definitions of I, M as an R-module and IM.
   ($\Rightarrow$) From the lecture, IM is defined as finite linear combinations
   $$\{\sum_{a\in A} i_a m_a | i_a \in I, m_a \in M\}.$$
   Consider $x = \sum_{u\in U} r_u u \in IM$. Then there exists i_a, m_a and an indexing set A such that $x = \sum_{a\in A} i_a m_a$ with $i_a \in I, m_a \in M$.
   Each term expands to the form
   $$i_a m_a = i_a \cdot \sum_{u\in U} (m_a)_u u = \sum_{u\in U} i_a (m_a)_u u$$
   where $(m_a)_u$ is in R (since it's the coefficient of u in the expansion over the basis U).
   That is, we can rewrite as
   $$\sum_{u\in U} r_u u = \sum_{u\in U} \sum_{a\in A} i_a (m_a)_u u.$$
   I is an ideal, so $i_a(m_a)_u \in I$ for all $a \in I$ and subsequently, for fixed u, we see that the finite sum $\sum_{a\in A} i_a(m_b)_u$ is also in I. We conclude (why can we conclude this?) that $\sum_{a\in A} i_a(m_b)_u = r_u$ and have that each r_u is in I.

Problem 9 (Free modules)

1. a) not free: one element cannot be a $\mathbb{Z}$-basis of $\mathbb{Q}$ (why?) and if the basis has at least two elements a/b and p/q there is a $\mathbb{Z}$-linear dependency $pb \cdot a/b + (-aq) \cdot p/q = 0$

b) free: the map $\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$, $(m, n) \mapsto 2m-n$ is surjective and has kernel $(m, 2m)$, $m \in \mathbb{Z}$. By the isomorphism theorem, $(\mathbb{Z} \oplus \mathbb{Z})/\langle(1, 2)\rangle \cong \mathbb{Z}$.

c) not free: $2 \cdot (1, 1) = 0$, but a free $\mathbb{Z}$-module has no elements of finite order (why?).

2. Let M be an R-module. By the universal property of free modules, the identity map $M \to M$ induces the R-module homomorphism $f : R^{(M)} \to M$, $(r_m)_{m\in M} \mapsto \sum_{m\in M} r_m.m$, which is surjective. By the isomorphism theorem, $R^{(M)}/\ker(f) \cong M$.

3. By the universal property of free modules, we know for all $f \in \mathrm{Map}(S, M)$, there is a unique $f_* \in \mathrm{Hom}_R(R^{(S)}, M)$ extending the diagram, i.e. with $f = f_* \circ \iota$. That makes the natural candidate for an inverse map $f_* \mapsto f_* \circ \iota$. Composition both directions is clearly identity. We need to show we have that $f \mapsto f_*$ is an abelian group homomorphism. For $f, g \in \mathrm{Map}(S, M)$, $(f + g)(s) := f(s) + g(s)$. Then

$$(f + g)_*(\sum_{s\in S} r_s s) := \sum_{s\in S} r_s(f + g)(s) = \sum_{s\in S} r_s(f(s) + g(s))$$

and since this is a finite sum, this is equal to $\sum_{s\in S} r_s f(s)+\sum_{s\in S} r_s g(s)$ and so $(f + g)_* = f_* + g_*$.

Write $\phi_M : \mathrm{Map}(S, M) \to \mathrm{Hom}_R(R^{(S)}, M)$ for the isomorphism. Given $h : M \to N$ let $h' : \mathrm{Map}(S, M) \to \mathrm{Map}(S, N)$ be given by $h'(f) = h \circ f$ and similarly $h'' : \mathrm{Hom}_R(R^{(S)}, M) \to \mathrm{Hom}_R(R^{(S)}, N)$ by $h''(g) = h \circ g$. The following diagram commutes:

$$\begin{tikzcd}
\mathrm{Map}(S, M) \arrow[r, "\phi_M"] \arrow[d, "h'"] & \mathrm{Hom}_R(R^{(S)}, M) \arrow[d, "h''"] \\
\mathrm{Map}(S, M) \arrow[r, "\phi_N"] & \mathrm{Hom}_R(R^{(S)}, N)
\end{tikzcd}$$

Indeed, let $f \in \mathrm{Map}(S, M)$. Then commutativity of the diagram amounts to $(h \circ f)_* = h \circ f_*$. To verify the latter identity, it is enough to check it on the basis S: $(h \circ f)_*(s) = h(f(s))$ and $(h \circ f_*)(s) = h(f(s))$.

Problem 19 (Bases with different cardinality)

1. Define maps ψ_1, ψ_2 as in the hint. Then

$$\psi_1\varphi_1+\psi_2\varphi_2 = (a_1, 0, a_3, 0 \ldots)+(0, a_2, 0, a_4 \ldots) = (a_1, a_2, a_3, a_4, \ldots) = \mathrm{id}_M$$

i.e. is the identity map on M and, so, for any $r \in R$, we have that $r = r\psi_1\varphi_1 + r\psi_2\varphi_2$. That is, $R = R\varphi_1 + R\varphi_2$ and so $\{\varphi_1, \varphi_2\}$ is a generating set.

To show linear independence, we use the identities as suggested in the hint (why do they hold?). Let $r_1\varphi_1 + r_2\varphi_2 = 0$. Then also $r_1\varphi_1\psi_1 + r_2\varphi_2\psi_1 = 0$, and by the identities, $r_1 = 0$. That $r_2 = 0$ follows analogously.

2. By part (a), the map $A : R^2 \to R$ given by $(r_1, r_2) \to r_1\varphi_1 + r_2\varphi_2$ is an isomorphism of R-modules. This yields $R \cong R^n$ by induction. (What is the induction step?)

Problem 11 (Opposite category)

We need a map $\circ^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}}(B, C) \times \mathcal{C}^{\mathrm{op}}(A, B) \to \mathcal{C}^{\mathrm{op}}(A, C)$, i.e. a map $\mathcal{C}(C, B) \times \mathcal{C}(B, A) \to \mathcal{C}(C, A)$. We take $f \circ^{\mathrm{op}} g := g \circ f$. The units are then simply $1_A^{\mathrm{op}} = 1_A$. Associativity and the unit property are immediate (details?).