Hints and solutions for problem sheet # 02

Advanced Algebra — Winter term 2016/17

(Ingo Runkel)

Problem 4

1. Take the indexing set $I = \mathbb{N}$, take $R = \mathbb{Z}$, $N = \mathbb{Z}$ and all $M_i = \mathbb{Z}$. Choose all $f_i : N \to M_i$ to be the identity. The element $f(1) \in \bigoplus_i M_i$ would need to satisfy $p_i(f(1)) = 1$, i.e. $f(1)$ must be the tuple $(1, 1, 1, \dots)$. But this has infinitely many entries which are non-zero and is hence not in the direct sum.

2. With the same choices for R and M_i as above, write $P = \prod_i M_i$ and $S = \bigoplus_i M_i$. Then $S \subset P$ is a submodule (why?) and we can consider the R-module map given by the canonical projection $\pi : P \to P/S$. Note that P/S is not zero, as e.g. $(1, 1, 1, \dots)$ is in P but not in S. Hence the map π is not zero (as it is surjective).

   Take $N = P/S$ and take all maps $g_i : N \to M_i$ to be zero. Then the two diagrams

$$\begin{tikzcd} \prod_i M_i \arrow[rr, "0"] & & N \\ & M_i \arrow[ul, "e_i"] \arrow[ur, "0"'] & \end{tikzcd} \quad , \quad \begin{tikzcd} \prod_i M_i \arrow[rr, "\pi"] & & N \\ & M_i \arrow[ul, "e_i"] \arrow[ur, "0"'] & \end{tikzcd}$$

   commute for all i. (Why does the right diagram commute?) Thus there is not a unique map making the diagrams commute.

Problem 5

1. Clear.

2. The bimodule structure is as follows. Let $f \in \mathrm{Hom}_R(M, N)$. Then, for $s \in S$, $t \in T$, $(s.f)(m) := f(m.s)$ and $(f.t)(m) := f(m).t$. Note that $m \in {}_RM_S$ and $f(m) \in {}_RN_T$, so the two right-actions in the defining equations do indeed make sense. Bilinearity is clear, as is that the two actions commute. For associativity, e.g.

$$(ss'.f)(m) = f(m.(ss')) = f((m.s).s') = (s'.f)(m.s) = (s.(s'.f))(m) \; .$$

Problem 6

1. We treat u as an example. Consider the map $f : \mathbb{C}[X] \to M_b$, $f(X^n) = X^{n+b-a} + \langle X^b \rangle$. This is an R-module homomorphism (why?). Then $f(X^a) = X^{a+b-a} + \langle X^b \rangle = 0$, and so f maps the submodule $\langle X^a \rangle$ to zero. By Proposition 2.1.2 we obtain an R-module map $f_* : M_a \to M_b$ which satisfies $f_*(X^n + \langle X^a \rangle) = X^{n+b-a} + \langle X^b \rangle$.

2. The kernel of the R-module homomorphism $\mathbb{C}[X] \to M_b$, $p \mapsto pX^{b-a}$ is $\langle X^a \rangle$ (why?). Hence the map $u : M_a \to M_b$ is injective.

   The map v is clearly surjective.

   The kernel of v is $\langle X^{b-a} \rangle$, which is precisely the image of u.

   If $a = 0$ or $a = b$, one of the modules is 0 and the seqences splits (via the zero map).

   For $0 < a < b$, the sequence does not split. Note that $u(1) = X^{b-a} + \langle X^b \rangle$. By Proposition 2.3.1, it is enough to show that there is no R-module homomorphism $\varphi : M_b \to M_a$ such that $\varphi(X^{b-a} + \langle X^b \rangle) = 1 + \langle X^a \rangle$. Consider $\varphi(1 + \langle X^b \rangle)$ and write $\varphi(1 + \langle X^b \rangle) = p + \langle X^a \rangle$ for some $p \in \mathbb{C}[X]$. Let d be the degree of the smallest non-zero monomial in p. Then $\varphi(X^{b-a} + \langle X^b \rangle) = X^{b-a}p + \langle X^a \rangle$ (why?), and the degree of the smallest non-zero monomial in $X^{b-a}p$ is $d + b - a > 0$. Hence $\varphi(X^{b-a} + \langle X^b \rangle) \neq 1 + \langle X^a \rangle$.

Problem 7

1. Apply Cor. 2.1.3 to to $g : M \to N$ to get that $\mathrm{im}(g) \cong M/\ker(g)$. By exactness, we know that g is surjective, so $\mathrm{im}(g) \cong N$ and $\ker(g) \cong \mathrm{im}(f)$, i.e.
$$N \cong \mathrm{im}(g) \cong M/\ker(g) \cong M/\mathrm{im}(f).$$
   Similarly, $L \cong \mathrm{im}(f) = \ker(g)$.

2. a) $\Rightarrow$ b). First we show that $\ker(f^*) \subset \mathrm{im}(g^*)$. Consider $k \in \ker(f^*)$. Then $0 = f^*(k) = kf$, and hence $0 = k(\mathrm{im} f) = k(\ker g)$. By (5 ii)), k induces a homomorphism $\bar{k} : B/\ker(g) \to M$ such that $\bar{k}(b + \ker(g)) = k(b)$. By (5 ii)) again, there's an isomorphism $\phi : B/\ker(g) \cong C$ such that $\phi(b + \ker(g)) = g(b)$. Then the map $\bar{k}\phi^{-1} : C \to M$ is an R-mod hom such that $g^*(\bar{k}\phi^{-1}) = k$. This shows that $\ker(f^*) \subset \mathrm{im}(g^*)$. The rest of this half of the proof is analogous to the one for 1) $\Rightarrow$ 2).

   b) $\Rightarrow$ a). Choose $M = C/\mathrm{im}(g)$ and let $\pi : C \to M$ be the canonical projection. Then $g^*(\pi) = \pi g = 0$ and $\ker g^* = 0$ implies $\pi = 0$ and hence $C = \mathrm{im}(g)$ and $B \xrightarrow{g} C \to 0$ is exact. Similarly, we show that $\ker(g) \subset \mathrm{im}(f)$ by letting $M = B/\mathrm{im}(f)$ and considering the canonical projection $B \to M$. Finally, let $M = C$, then $0 = f^*g^* = gf$, so imim$f \subset \ker(g)$. Thus, $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact.