Problem sheet # 11 Advanced Algebra Winter term 2016/17

(Ingo Runkel)

Problem 45 (Missing steps in proof of Theorem 5.2.1)
Show the implications $1 \Rightarrow 3$ and $3 \Rightarrow 2$ in Theorem 5.2.1.

Problem 46 (Examples and counter examples)

1. Give an example showing that submodules of projective modules need not be projective.

   Hint: Try to split $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ (p prime), all considered as modules over $\mathbb{Z}/p^2\mathbb{Z}$.

2. Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

3. Let $M \neq 0$ be a finitely generated module over some ring. Is there always / sometimes / never a finitely generated injective module J with $M \subset J$?

Problem 47 (Projective, injective, semisimple)
Let R be a ring. Show that the following are equivalent:

1. Every R-module is projective.
2. Every R-module is injective.
3. Every R-module is semisimple.
4. Every simple R-module is projective.

Extra (and not so easy) problem without points: The statement "Every simple R-module is injective." is missing from the above list. Can you say why?

Please turn over.

Problem 48 (Universal property of pullbacks)

For an arbitrary category C and morphisms $f : A \to X$, $g : B \to X$ in C, a *pullback* of (f, g) is a pair of morphisms $\alpha : Y \to A$, $\beta : Y \to B$ such that $f \circ \alpha = g \circ \beta$ satisfying the following universal property: given $\gamma : Z \to A, \delta : Z \to B$ with $f \circ \gamma = g \circ \delta$ there exists a unique $\xi : Z \to Y$ with $\gamma = \alpha \circ \xi$, $\delta = \beta \circ \xi$:

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists!\xi"] \arrow[drr, bend left, "\gamma"] \arrow[ddr, bend right, "\delta"'] & & \\
& Y \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & X
\end{tikzcd}$$

Given R-module homomorphisms $B \xrightarrow{g} M \xleftarrow{f} A$, in the lecture we defined an R-module M' which we called the pullback:

$$M' := \big\{(a, b) \in A \oplus B \,\big|\, f(a) = g(b)\big\} \ .$$

Show that M' satisfies the universal property of pullbacks.

Problem 49 (A diagram chase)

Let R be a commutative ring and consider the following a commutative diagram of R-modules and homomorphisms with exact rows:

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "t_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "t_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "t_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "t_4"] & A_5 \arrow[d, "t_5"] \\
B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5
\end{tikzcd}$$

Prove one of the two following claims (or both, if you like):

1. If t_2 and t_4 are surjective and t_5 is injective, then t_3 is surjective.
2. If t_2 and t_4 are injective and t_1 is surjective, then t_3 is injective.

Remark: It follows from parts 1 and 2 that if t_1 is surjective, t_2 and t_4 are isomorphisms and t_5 is injective, then t_3 is an isomorphism.