

Problem sheet # 11 Advanced Algebra Winter term 2016/17

(Ingo Runkel)

Problem 45 (Missing steps in proof of Theorem 5.2.1)

Show the implications $1 \Rightarrow 3$ and $3 \Rightarrow 2$ in Theorem 5.2.1.

Problem 46 (Examples and counter examples)

1. Give an example showing that submodules of projective modules need not be projective.

Hint: Try to split $\mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ (p prime), all considered as modules over $\mathbb{Z}/p^2\mathbb{Z}$.

2. Show that \mathbb{Q} is not a projective \mathbb{Z} -module.
3. Let $M \neq 0$ be a finitely generated module over some ring. Is there always / sometimes / never a finitely generated injective module J with $M \subset J$?

Problem 47 (Projective, injective, semisimple)

Let R be a ring. Show that the following are equivalent:

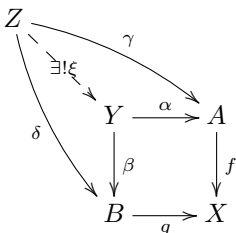
1. Every R -module is projective.
2. Every R -module is injective.
3. Every R -module is semisimple.
4. Every simple R -module is projective.

Extra (and not so easy) problem without points: The statement “Every simple R -module is injective.” is missing from the above list. Can you say why?

Please turn over.

Problem 48 (Universal property of pullbacks)

For an arbitrary category C and morphisms $f : A \rightarrow X, g : B \rightarrow X$ in C , a *pullback* of (f, g) is a pair of morphisms $\alpha : Y \rightarrow A, \beta : Y \rightarrow B$ such that $f \circ \alpha = g \circ \beta$ satisfying the following universal property: given $\gamma : Z \rightarrow A, \delta : Z \rightarrow B$ with $f \circ \gamma = g \circ \delta$ there exists a unique $\xi : Z \rightarrow Y$ with $\gamma = \alpha \circ \xi, \delta = \beta \circ \xi$:



Given R -module homomorphisms $B \xrightarrow{g} M \xleftarrow{f} A$, in the lecture we defined an R -module M' which we called the pullback:

$$M' := \{(a, b) \in A \oplus B \mid f(a) = g(b)\} .$$

Show that M' satisfies the universal property of pullbacks.

Problem 49 (A diagram chase)

Let R be a commutative ring and consider the following a commutative diagram of R -modules and homomorphisms with exact rows:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow t_1 & & \downarrow t_2 & & \downarrow t_3 & & \downarrow t_4 & & \downarrow t_5 \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

Prove one of the two following claims (or both, if you like):

1. If t_2 and t_4 are surjective and t_5 is injective, then t_3 is surjective.
2. If t_2 and t_4 are injective and t_1 is surjective, then t_3 is injective.

Remark: It follows from parts 1 and 2 that if t_1 is surjective, t_2 and t_4 are isomorphisms and t_5 is injective, then t_3 is an isomorphism.