Problem sheet # 10 Advanced Algebra Winter term 2016/17

(Ingo Runkel)

Problem 40 (Tensor product over quotient rings)

Prove Lemma 4.5.4: Let P be a ring, $I \subset P$ an ideal and $R := P/I$. R-modules M_R, ${}_RN$ can be understood as P-modules via the ring-homomorphism provided by the canonical projection $\pi : P \to R$. Then

$$M \otimes_R N \ = \ M \otimes_P N \ .$$

Problem 41 (Tensor product of bimodules)

Prove part of Proposition 4.5.7: Let R, S, T be rings and ${}_RM_S$, ${}_SN_T$ be bimodules.

1. Let $r \in R$. Show that there exists a unique $\mathbb{Z}$-module homomorphism

   $$\rho(r) : M \otimes_S N \to M \otimes_S N$$

   such that $\rho(r)(m \otimes_S n) = (r.m) \otimes_S n$ for all $m \in M$, $n \in N$.

2. Show that $r \mapsto \rho(r)$ turns $M \otimes_S N$ into an R-left module.

3. Let ${}_RM'_S$ be another R-S-bimodule and $f : M \to M'$ a bimodule homomorphism. Show that $f \otimes_S id_N : M \otimes_S N \to M' \otimes_S N$ is a R-left module homomorphism.

Problem 42 (Tensor product of left modules)

Let R be a commutative ring.
Denote by ι the functor R-Mod $\to$ R-Mod-R as defined in Section 4.5. Write U for the forgetful functor from R-R-bimodules to R-left modules. In Section 4.5 we defined the tensor product of two R-left modules ${}_RM, {}_RN$ as

$$M \otimes_R N \ := \ U\big(\, \iota(M) \otimes_R \iota(N)\,\big) \ .$$

1. Give an example of an R-R-bimodule module X such that $\iota(U(X)) \neq X$.

2. Show that $\iota(U(\iota(M) \otimes_R \iota(N))) = \iota(M) \otimes_R \iota(N)$, i.e. the bimodule structure of $\iota(M) \otimes_R \iota(N)$ can be reconstructed from its left module structure.

Please turn over.

Problem 43 (Tensor product via R-bilinear maps)

Let R be a commutative ring.

Let M, N, L be R-modules. A map $\beta : M \times N \to L$ is called R-bilinear if $\beta(m+m', n) = \beta(m, n)+\beta(m', n)$, $\beta(m, n+n') = \beta(m, n)+\beta(m, n')$, as well as $\beta(r.m, n) = r.\beta(m, n)$ and $\beta(m, r.n) = r.\beta(m, n)$ (for all $m, m' \in M$, $n, n' \in N$ and $r \in R$).

Given R-modules M, N, show:

1. The map $\otimes_R : M \times N \to M \otimes_R N$ is R-bilinear.

2. The pair $(M \otimes_R N, \otimes_R)$ (as defined in the lecture or in Problem 42 and part 1) also satisfies the following universal property: For every R-module L and R-bilinear map $\beta : M \times N \to L$, there exists a unique R-module homomorphism $\tilde{\beta} : M \otimes_R N \to L$ such that $\beta = \tilde{\beta} \circ \otimes_R$.

Remark: For R a field, the above notion of an R-bilinear map is the usual definition of a bilinear map from a pair of vector spaces to a vector space. The universal property in 2 is used to define the tensor product of two vector spaces over a field. While the explicit construction for our tensor product over rings (as a quotient of two $\mathbb{Z}$-modules) and of vector spaces (as a quotient of two vector spaces) is different, by part 2 are isomorphic via a unique isomorphism compatible with $\otimes_R$.

Problem 44 (Tensor products over $\mathbb{Z}$)

Let R be a ring, and $I \subset R$ a (two-sided) ideal. Show:

1. For each R-left module M, there is a R-module isomorphism $f : R/I \otimes_R M \to M/IM$ such that f maps $(r + I) \otimes m$ to $r.m + IM$.

2. Let $J \subset R$ be a left ideal. Show that as abelian groups, $R/I \otimes_R R/J \cong R/(I + J)$.

3. Let $m, n \in \mathbb{N}$ and let d be the greatest common divisor of m and n. Show that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$.