Problem sheet # 09 Advanced Algebra
Winter term 2016/17

(Ingo Runkel)

Problem 35 (Directness of infinite sums)

Let $(M_i)_{i \in I}$ be a family of submodules of an R-module M. Show that the following statements are equivalent:

1. The sum of the M_i over all $i \in I$ is direct.
2. For every finite subset F of I, the sum of the M_i over $i \in F$ is direct.

Problem 36 (Semisimplicity and exact sequences)

Let $L \to M \to N$ be a short exact sequence of R-modules. Show that, if M is semisimple, so are L and N. What can you say about the converse statement?

Remark: This proves Corollary 3 to Theorem 4.4.1 (why?).

Problem 37 (Semisimplicity of group rings)

Let K be a field and G a finite group such that the characteristic of K does not divide the order of G (or, equivalently, under the canonical ring homomorphism $\mathbb{Z} \to K$, $|G|$ is not zero).

Prove that the group ring $K[G]$ is semi-simple. (See Problem 1 for the definition of $K[G]$.)

Hint: Let V be a $K[G]$-module and U a submodule of V. As *K-vector spaces*, one can always find a complement W in V such that $V = U \oplus W$. Let P be projection onto U. Note that in general, P is not a $K[G]$-module homomorphism. Let

$$\bar{P} := \frac{1}{|G|} \sum_{g \in G} \rho(g) P \rho(g^{-1})$$

where $\rho(g)$ is the action of g on V. Show 1) $\bar{P}$ is a $K[G]$-module homomorphism, and 2) $\bar{P}^2 = \bar{P}$.

Please turn over.

Problem 38 (Wedderburn-Artin for algebras)
Let K be a field.

1. Let R, S be rings. Show that if $R \times S$ is a K-algebra, so are R and S.

2. Let R be a ring. Show that the centre of the matrix ring $\mathrm{Mat}_n(R)$ is

$$Z(\mathrm{Mat}_n(R)) \; = \; Z(R) \cdot I_{n \times n} \; .$$

   Show that if $\mathrm{Mat}_n(R)$ is a K-algebra, so is R.

3. Let K be a field. A K-algebra is called semisimple if it is semisimple as a ring. Formulate and prove the Wedderburn-Artin Theorem for K-algebras.

4. List all 9-dimensional semisimple $\mathbb{R}$-algebras up to isomorphism.

Problem 39 (Semisimplicity and infinite products)
Let I be an infinite set and $(R_i)_{i \in I}$ a family of non-zero rings. The direct sum $\bigoplus_{i \in I} R_i$ of the R_i is a non-unital associative ring. But the direct product $S := \prod_{i \in I} R_i$ is again a unital associative ring (i.e. a ring in our use of the term).

Suppose all R_i are semisimple. Is S then always semisimple? Or sometimes semisimple? Or never semisimple?

Additional problem without points: Consider the abelian group $G = \mathbb{R}/\mathbb{Z}$ (this is a compact abelian Lie group, often called $U(1)$ or $SO(2)$).

1. The group ring $\mathbb{C}[G]$ possesses an infinite number of mutually non-isomorphic one-dimensional (over $\mathbb{C}$) modules. Can you find an infinite number of these modules?

2. Since one-dimensional modules are automatically simple, this gives an infinite number of isomorphism classes of simple modules. So by part 1, $\mathbb{C}[G]$ cannot be semisimple and there must be a submodule $I \subset \mathbb{C}[G]$ such that there is no submodule J with $I \oplus J = \mathbb{C}[G]$. Can you find an example?

3. Giving a $\mathbb{C}$-linear representation of G is the same as giving a representation of $\mathbb{C}[G]$ (why?). You may have heard a statement like "Representations of compact Lie groups are semisimple." – Is this not a contradiction?