Problem sheet # 08 Advanced Algebra
Winter term 2016/17

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Problem 30 (Composition series and exact sequences)

Let $L \xrightarrow{f} M \xrightarrow{g} N$ be a short exact sequence of R-modules.

1. Let $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_r = 0$ and $N = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_s = 0$ be composition series. Use these to obtain a composition series of M and show that $l(M) = l(L) + l(N)$.

2. Show that for a exact sequence $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ of finite length R-modules, $\sum_{i=0}^{n}(-1)^i l(M_i) = 0$.

Problem 31 (Trying to generalise Jordan-Hölder)

Let R be a ring and M an R-module. The Jordan-Hölder Theorem states that if M has finite length, all of its composition series are equivalent. Here are two "half-infinite generalisations of a composition series" one can try to consider: Let M_i, $i \in \mathbb{Z}_{\geq 0}$ submodules of M such that either

1. $M = M_0 \supset M_1 \supset M_2 \supset \dots$, where $\bigcap_{i=1}^{\infty} M_i = \{0\}$ and M_i/M_{i+1} is simple, or

2. $\{0\} = M_0 \subset M_1 \subset M_2 \subset \dots$, where $\bigcup_{i=1}^{\infty} M_i = M$ and M_{i+1}/M_i is simple.

Give a counter example to the "generalised Jordan-Hölder Theorem" for the half-infinite composition series in 1.

Additional problem without points: What happens in attempt 2?

Problem 32 (Modules with inequivalent indecomposable decompositions)

1. Let R be a ring and I, J left ideals in R such that $R = I + J$. Show that $R \oplus (I \cap J) \cong I \oplus J$ as R-modules.

2. We now use the above to construct an example of two inequivalent indecomposable decompositions of a module. Let $R = \mathbb{Z}[\sqrt{-5}]$, i.e. the subring $R = \left\{\, m + n\sqrt{-5} \,\middle|\, m, n \in \mathbb{Z} \,\right\}$ of $\mathbb{C}$. For $r \in R$ write $|r|$ for the norm in $\mathbb{C}$, so that $|rs| = |r||s|$ for $r, s \in R$.

   (a) Show that $I = \langle 3\,,\, 2+\sqrt{-5}\rangle$ and $J = \langle 3\,,\, 2-\sqrt{-5}\rangle$ are not principal ideals (i.e. they cannot be generated by a single element).

   (b) Show that $I + J = R$, and that R is not isomorphic to I or J.

Please turn over.

Problem 33 (Modules without indecomposable decomposition)

Let R be continuous functions from $\mathbb{Q}$ to $\mathbb{R}$. Show that ${}_R R$ does not have an indecomposable decomposition.

Hint: You could proceed as follows.

1. Show that a nonzero $f \in R$ is nonzero at at least two points.
2. Show that for $a \in \mathbb{R}$ irrational, the set $U_{<a} := (-\infty, a) \cap \mathbb{Q}$ is open and closed in $\mathbb{Q}$. Hence the characteristic function $\chi_{U_{<a}}$ on $U_{<a}$ which is 1 on $U_{<a}$ and 0 elsewhere is continuous as a function $\mathbb{Q} \to \mathbb{Z}$.
3. To show there is no indecomposable decomposition, show that each ideal has a decomposition into submodules.

Problem 34 (Indecomposable modules over polynomial rings)

1. Give all finite length indecomposable modules over $\mathbb{C}[X]$ up to isomorphism.

   Hint: We know all simple $\mathbb{C}[X]$ modules. Conclude that finite length modules are finite dimensional as $\mathbb{C}$-vector spaces. Use the Jordan normal form.
2. Give composition series for all modules in part 2. What are their length and what their composition factors?