Problem sheet # 07 Advanced Algebra
Winter term 2016/17

(Ingo Runkel)

Problem 26 (not artinian, not noetherian)

1. Show that $\mathbb{Q}$ is neither noetherian nor artinian as a $\mathbb{Z}$-module.
2. For a prime p let $Q_p = \{m/p^n \in \mathbb{Q}/\mathbb{Z} \,|\, m \in \mathbb{Z}, n \geq 0\}$. Show that Q_p is artinian as a $\mathbb{Z}$-module. What about $\mathbb{Q}/\mathbb{Z}$ itself?

Problem 27 (left and right noetherian / artinian are different)
Let S be a ring and $T \subset S$ be a subring. Define

$$R = \left\{ \begin{pmatrix} s & s' \\ 0 & t \end{pmatrix} \,\middle|\, s, s' \in S \ , \ t \in T \right\} .$$

R is a ring via matrix addition and multiplication. Suppose that the left modules ${}_SS$ and ${}_TT$ are noetherian (resp. artinian), but that the right T-module S_T is not noetherian (resp. artinian).

1. Give examples of rings S, T satisfying the above condition (one example for each case).
2. Show that R is left noetherian (resp. artinian) but not right noetherian (resp. artinian).

Problem 28 (injective and surjective maps for noetherian modules)
Let R be a ring and let N be a noetherian R-module. Let M be an R-module such that there is an injective R-module homomorphism $N \to M$.

1. Show that a surjective R-module homomorphism $N \to M$ is automatically an isomorphism.

   Hint: Think of N as a submodule of M. Let $X_n := \{x \in N \,|\, f^k(x) \in N \text{ for } k = 1, \dots, n-1 \text{ and } f^n(x) = 0\}$. This is an ascending chain of submodules. Why can one write $y \in \ker(f)$ as $y = f^n(z_n)$ for any n and appropriate $z_n \in N$?

2. Does the statement of part 1 still hold if N is artinian?

Please turn over.

Problem 29 (Simple modules over polynomial rings)

1. In Section 4.2 we discussed simple $K[X]$ modules for an algebraically closed field K. We defined $K_\lambda := K[X]/(X - \lambda)$ for $\lambda \in K$.

   Show that $\dim_K K_\lambda = 1$, and that K_λ and K_μ are isomorphic iff $\lambda = \mu$.

2. Let K be a field. Consider $K[X_1, X_2, \ldots, X_n]$ and let $\lambda_1, \ldots, \lambda_n \in K$. Show that $\langle X_1 - \lambda_1, \ldots, X_n - \lambda_n \rangle$ is a maximal ideal.