Problem sheet # 06 Advanced Algebra
Winter term 2016/17

(Ingo Runkel)

Problem 21 (The centre of a ring as natural endomorphisms)

Let R be a ring and let Id be the identity functor on R-Mod. Show that there is a one-to-one correspondence between elements of $Z(R)$ and natural transformations $Id \to Id$.

Hint: It is easy to assign a natural transformation to an element of the centre. The converse is more work. You could proceed in two steps. 1) Extract the central element by what the natural transformation does to R, and 2) find enough R-module maps $R \to M$ to fix the natural transformation on M in terms of its value on R.

Problem 22 (Dual vector spaces)

Let k be a field and recall the contravariant functor $(-)^* : k\text{-Mod} \to k\text{-Mod}$. The question "Is the functor of taking duals naturally isomorphic to the identity functor?" makes no sense as one functor is contravariant and the other covariant.

But suppose you did not know about functors. You could ask yourself "Can I find a family of isomorphisms $U \to U^*$ for all finite-dimensional k-vector spaces U which behaves well with respect to linear maps $V \to W$?". Make this precise and show that the answer is "No."

Problem 23 (A category of matrices)

Let k be a field and let $\mathcal{N}$ be the category with objects $\mathbb{Z}_{\geq 0}$ and morphism sets $\mathcal{N}(m,n) = \mathrm{Mat}_{n\times m}(k)$. Show that $\mathcal{N}$ is equivalent to $k\text{-Mod}_{\mathrm{fin}}$, the category of finite-dimensional k-vector spaces.

Problem 24 (Details for the free modules / forget adjunction)

Let A be an abelian group and R a ring.

1. Consider the map $\psi_{M,A} : \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R,A)) \to \mathrm{Hom}_{\mathbb{Z}}(M,A)$ which assigns to γ the map $m \mapsto (\gamma(m))(1)$. Show that $\psi_{M,A}$ is a bijection.
2. Show that $\psi_{M,A}$ is natural in M and A

Problem 25 (From the proof of Proposition 3.6.3)

Let $G : \mathcal{D} \to \mathcal{C}$ be a functor, and let F and F' be two left adjoints with unit, counit η, ε and η', ε', respectively. Show that

$$F \circ Id \xrightarrow{F\eta'} FGF' \xrightarrow{\varepsilon F'} Id \circ F'$$

is a natural isomorphism.