Problem sheet # 05 Advanced Algebra Winter term 2016/17

(Ingo Runkel)

Problem 17 (Additive but not abelian)

Let k be a field. A $\mathbb{Z}$-*filtered vector space* is a k-vector space V together with subspaces $(V_i)_{i\in\mathbb{Z}}$, such that $V_i \subset V_{i+1}$ for all i. Let $\mathcal{C}$ be the category of $\mathbb{Z}$-filtered vector spaces. The objects of $\mathcal{C}$ are $\mathbb{Z}$-filtered vector space and the morphisms are linear maps $f : V \to W$ such that $f(V_i) \subset W_i$ for all i. You may assume (or prove, if you like) that $\mathcal{C}$ is additive.

Show that

1. $\mathcal{C}$ has kernels and cokernels.
2. $\mathcal{C}$ is not abelian.

   Hint: Consider $V = W = k$ with $V_i = W_i = 0$ for $i < 0$ and $V_i = W_i = k$ for $i > 0$. Choose $V_0 = 0$ and $W_0 = k$. Let $f : V \to W$ be the identity map on k as a map from V to W. Show that f is mono and epi but not an isomorphism.

Problem 18 (Rings and modules as categories and functors)

An *Ab-category* is a category whose morphism spaces are abelian groups and whose composition is bilinear. An *additive* functor between two Ab-categories is defined the same way as for additive categories: the maps between morphism spaces are group homomorphisms.

1. Let $\mathcal{C}$ be an Ab-category and $A \in \mathcal{C}$. Show that $\mathcal{C}(A, A)$ is a ring. Make sense of: "An Ab-category with one object is the same as a ring". (Why does this not work for additive categories?)
2. Let now $\mathcal{C}$ be an Ab-category with one object, which we will call $\bullet$. Write $R = \mathcal{C}(\bullet, \bullet)$ for the corresponding ring. Make sense of: "An additive functor $\mathcal{C} \to \mathbf{Ab}$ is the same as an R-module."
3. Let $\mathcal{C}$ be as in 2. Make sense of "An natural transformation between two additive functors $\mathcal{C} \to \mathbf{Ab}$ is the same as an R-module homomorphism."

Please turn over.

Problem 19 (Bifunctors)

1. Make precise the notion of the product category $\mathcal{C} \times \mathcal{D}$ of two categories $\mathcal{C}, \mathcal{D}$ (objects are pairs – explain morphisms and composition).

2. A functor from a product category $\mathcal{C} \times \mathcal{D}$ to another category $\mathcal{E}$ is called a bifunctor from $(\mathcal{C}, \mathcal{D})$ to $\mathcal{E}$. Show that, for each $C \in \mathcal{C}$ and $D \in \mathcal{D}$, a bifunctor F determines functors $F_C : \mathcal{D} \to \mathcal{E}$ and $F_D : \mathcal{C} \to \mathcal{E}$ such that for any morphisms $f \in \mathcal{C}(C, C'), g \in \mathcal{D}(D, D')$ the following diagram commutes:

$$\begin{tikzcd} F(C,D) \arrow[r, "F_D(f)"] \arrow[d, "F_C(g)"'] & F(C',D) \arrow[d, "F_{C'}(g)"] \\ F(C,D') \arrow[r, "F_{D'}(f)"'] & F(C',D') \end{tikzcd}$$

3. Show that the converse of 2 is also true. That is, given a family of functors $F_C : \mathcal{D} \to \mathcal{E}$ and $F_D : \mathcal{C} \to \mathcal{E}$ (for all $C, D \in \mathcal{C}, \mathcal{D}$) that satisfy $F_C(D) = F_D(C)$ and a commuting square as above (what must $F(C, D)$ be?), they determine a bifunctor F, i.e. a functor from the product category $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$.

Problem 20 (Hom-functors are bifunctors)

Let $\mathcal{C}$ be a category and let $A \in \mathcal{C}$.

1. Show that $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$ is a functor.

2. Show that $\mathcal{C}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is a functor.

3. Show that $\mathcal{C}(-, -) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ is a bifunctor.