Problem sheet # 03 Advanced Algebra Winter term 2016/17

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Problem 8 (Misc)

1. Show that $\mathrm{End}_R({}_RR) \cong R^{\mathrm{op}}$ and $\mathrm{End}_{R\text{-}R}({}_RR_R) \cong Z(R)$ as rings. Here $\mathrm{End}_{R\text{-}R}({}_RR_R)$ are bimodule endomorphisms and $Z(R)$ is the centre of R.
2. Let R be a ring, I an ideal in R and M an R-module. Show that $IM = \{\sum_a i_a m_a | i_a \in I, m_a \in M\}$ is a submodule. Show that I is in the kernel of the representation homomorphism $\rho : R \to \mathrm{End}(M/IM)$.
3. Let R be a ring, I an ideal in R, M a free R-module with basis U. Show that $\sum_{u \in U} r_u u$ is in IM if and only if all r_u are in I.

Problem 9 (Free modules)

1. Consider the three $\mathbb{Z}$-modules a) $\mathbb{Q}$, b) $(\mathbb{Z} \oplus \mathbb{Z})/\langle(1,2)\rangle$, c) $(\mathbb{Z} \oplus \mathbb{Z})/\langle(2,2)\rangle$. Show in each case whether the module is free or not.
2. Show that every module is the quotient of a free module.
3. Show Cor. 2.4.2, that is, show that $\mathrm{Map}(S, M) \to \mathrm{Hom}_R(R^{(S)}, M)$, $f \mapsto f_*$, is an isomorphism of abelian groups. Can you think of a useful compatibility with R-module homomorphisms $h : M \to N$?

Problem 10 (Bases with different cardinality)

Let M be the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z} \times \cdots = \prod_{i \in \mathbb{Z}_{>0}} \mathbb{Z}$ (the direct product of $\mathbb{Z}$ over $i \in \mathbb{Z}_{>0}$). Let $R = \mathrm{End}_{\mathbb{Z}}(M)$. Define $\varphi_1, \varphi_2 \in R$ by

$$\varphi_1(a_1, a_2, a_3, \dots) = (a_1, a_3, a_5, \dots) \quad , \quad \varphi_2(a_1, a_2, a_3, \dots) = (a_2, a_4, a_6, \dots) \ .$$

1. Prove that $\{\varphi_1, \varphi_2\}$ is a free basis of the left R-module R.
   Hint: Define maps ψ_1 and ψ_2 by
   $$\psi_1(a_1, a_2, a_3, \dots) = (a_1, 0, a_2, 0 \dots) \ , \ \psi_2(a_1, a_2, a_3, \dots) = (0, a_1, 0, a_2 \dots) \ .$$
   Verify that $\varphi_i \psi_i = 1$, $\varphi_1 \psi_2 = 0 = \varphi_2 \psi_1$ and $\psi_1 \varphi_1 + \psi_2 \varphi_2 = id$.
2. Show that $R \cong R^2$ as R-modules and deduce that $R \cong R^n$ for all $n \in \mathbb{Z}_{>0}$.

Problem 11 (Opposite category)

Given a category $\mathcal{C}$, try to define a new category, the *opposite category* $\mathcal{C}^{\mathrm{op}}$ as follows: the objects of $\mathcal{C}^{\mathrm{op}}$ are those of $\mathcal{C}$. Given two objects $A, B \in \mathcal{C}^{\mathrm{op}}$, define $\mathcal{C}^{\mathrm{op}}(A, B) := \mathcal{C}(B, A)$. Can you define units 1_A^{op} and a composition $\circ^{\mathrm{op}}$ to turn $\mathcal{C}^{\mathrm{op}}$ into a category?