Problem sheet # 02 Advanced Algebra
Winter term 2016/17

(Ingo Runkel)

Problem 4 (Direct sums and products)

Recall the universal properties for $\bigoplus$, $\prod$ proved in Section 2.2. Let R be a ring and $(M_i)_{i \in I}$ be a family of R-modules.

1. Give an example to show that $\bigoplus_i M_i$ does in general not satisfy the universal property of the direct product.

   In more detail, find R, M_i, N and module homomorphisms $f_i : N \to M_i$ such that an $f : N \to \bigoplus_i M_i$ which makes the relevant diagrams commute either does not exist or exists but is not unique.

2. Give an example to show that $\prod_i M_i$ does in general not satisfy the universal property of the direct sum.

Problem 5 (Hom sets as modules)

Let R, S, T be rings and let ${}_R M_S$ be an R-S-bimodule and let ${}_R N_T$ be an R-T-bimodule. Consider the Hom set

$$H := \mathrm{Hom}_R(M, N) \ .$$

1. Show that H is an abelian group.
2. Find a way to turn H into a left S-module and into a right T-module, such that it becomes an S-T-bimodule.
3. Show that $\mathrm{Hom}_R({}_R R_R, {}_R N_T) \cong {}_R N_T$ as R-T-bimodules.

Problem 6 (Examples of short exact sequences)

Let $R = \mathbb{C}[X]$ and let $M_k = \mathbb{C}[X]/\langle X^k \rangle$, $k \geq 0$. From Def. & Prop. 2.1.1 we know that M_k is an R-module.

1. Show that for $0 \leq a \leq b$ there are well-defined R-module homomorphisms $u : M_a \to M_b$ and $v : M_b \to M_{b-a}$ which satisfy $u(X^n + \langle X^a \rangle) = X^{n+b-a} + \langle X^b \rangle$ and $v(X^n + \langle X^b \rangle) = X^n + \langle X^{b-a} \rangle$.
2. Show that for $0 \leq a \leq b$, $M_a \xrightarrow{u} M_b \xrightarrow{v} M_{b-a}$ is a short exact sequence. Show that for $0 < a < b$ this sequence does not split.

Problem 7 (Short exact sequences)

1. Let $L \xrightarrow{f} M \xrightarrow{g} N$ be a short exact sequence of R-modules. Show that $N \cong M/\operatorname{im}(f)$ and that $L \cong \ker(g)$ as R-modules.
2. Show part 2 in Thm. 2.3.2 (on the relation between exactness and Hom).