Problem sheet # 01 Advanced Algebra Winter term 2016/17

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In-class problems

1. Find an example of a ring R and an element $r \in R$ such that r has a right-inverse but not a left-inverse.

2. Show that a non-zero ring is a division ring iff every non-zero element has a right-inverse.

3. Find a left ideal in $\mathrm{Mat}_2(\mathbb{Q})$ which is not a right ideal.

4. Give an example of a ring homomorphism $f : R \to S$ such that $\mathrm{im}(f)$ is not an ideal in S and $\ker(f)$ is not a subring in R.

5. We defined a ring homomorphism $f : R \to S$ to be a map such that for all $a, b \in R$: $f(a + b) = f(a) + f(b)$, $f(ab) = f(a)f(b)$ and $f(1) = 1$. Show that this implies $f(0) = 0$.

6. Define a *non-unital ring homomorphism* $f : R \to S$ to be a map such that for all $a, b \in R$: $f(a+b) = f(a)+f(b)$, $f(ab) = f(a)f(b)$. Give an example of rings R, S (with unit) and a non-unital ring homomorphism such that $f(1) \neq 1$.

7. Show that if K is a field and R is any ring with $1 \neq 0$, then every ring homomorphism $K \to R$ is injective.

8. The *opposite ring* R^{op} of a ring R is equal to R as abelian group, but has product $r *_{\mathrm{op}} r' = r' * r$. Show that a right R-module M is the same as a left R^{op}-module (advice: use different symbols for the R and R^{op} actions, e.g. $m \cdot r$ and $r : m$).

9. Let R be a commutative ring. Show that $\mathrm{Mat}_n(R)^{\mathrm{op}} \cong \mathrm{Mat}_n(R)$ as rings.

10. Let $(R, +, \cdot, 0, 1)$ be a structure which obeys all properties of a (unital, associative) ring except for possibly that $a + b = b + a$ for all $a, b \in R$. Show that then already $a + b = b + a$ for all $a, b \in R$.

Please turn over.

Homework problems

Problem 1 (Monoid-rings)

Let M be a monoid (i.e. a set with an operation $\cdot : M \times M \to M$ which is associative and has a unit $1 \in M$, but not necessarily inverses). Let R be a ring. Then the *monoid ring* $R[M]$ is set of functions $M \to R$, such that $f(m) \neq 0$ for only finitely many $m \in M$, and with abelian group structure given by addition of functions and the convolution product as multiplication. In more detail, let χ_m be the characteristic function for $m \in M$, i.e. $\chi_m(n) = 0$ for $m \neq n$, and $\chi_m(m) = 1$. Then the elements of $R[M]$ are of the form $\sum_{m\in M} r_m \chi_m$ for some $r_m \in R$. The convolution product is the R-bilinear extension of $\chi_m * \chi_n = \chi_{m\cdot n}$. That is, for $f = \sum_{m\in M} r_m\chi_m$ and $g = \sum_{n\in M} s_n\chi_n$ with $r_m, s_n \in R$, one sets $f * g = \sum_{a,b\in M} r_a s_b \, \chi_{a\cdot b}$.

If M happens to be a group, $R[M]$ is also called *group ring*.

1. Show that for $f, g \in R[M]$ we have $(f * g)(m) = \sum_{a,b\in M, a\cdot b=m} f(a)g(b)$.
2. Show that $R[M]$ is indeed a ring. What is the (multiplicative) unit?
3. Consider $\mathbb{Z}_{\geq 0}$ as a semigroup wrt. $+, 0$. Show that $R[\mathbb{Z}_{\geq 0}]$ is isomorphic to the polynomial ring $R[X]$.
4. The *Quaternion group* Q_8 consists of 8 elements, $\{\pm 1, \pm i, \pm j, \pm k\}$. The product is uniquely determined by declaring 1 to be the unit, -1 to commute with all elements, $(-1)^2 = 1$, and $i^2 = j^2 = k^2 = ijk = -1$. The group ring $\mathbb{R}[Q_8]$ is an eight-dimensional algebra over $\mathbb{R}$. The elements $\{\pm 1, \pm i, \pm j, \pm k\}$ all have multiplicative inverses. Why is it not a division algebra over $\mathbb{R}$? (Show this explicitly, do not just use Frobenius' Theorem.)

Problem 2 (Representations of polynomial rings)

Show that giving a module M over the polynomial ring $R[X_1, \dots, X_n]$ is equivalent to giving endomorphisms $\alpha_1, \dots, \alpha_n \in \mathrm{End}_R(M)$ such that $\alpha_i\alpha_j = \alpha_j\alpha_i$ for all i, j.

Problem 3 (Representations vs. modules)

A *representation* of a ring R is an abelian group M together with a ring homomorphism $\rho : R \to \mathrm{End}(M)$.

Let R be a ring and let M be an abelian group. Consider the two sets

$$A = \{\, \sigma : R \times M \to M \mid (M, \sigma) \text{ is a left } R\text{-module} \,\} ,$$
$$B = \{\, \rho : R \to \mathrm{End}(M) \mid (M, \rho) \text{ is a representation of } R \,\} .$$

1. Find a bijection between these two sets. ("Modules and representations are the same thing.")
2. How can the notion of module homomorphisms be transported to representations, i.e. what is a homomorphism of representations? How does your bijection behave with respect to homomorphisms of modules / representations?