Reconstruction, Pt. 1

Vincent Koppen 27th January, 2017

Let k be a field and let H be a Hopf algebra over k.

In this talk we consider, instead of the Hopf algebra H itself, the category of all its representations, and examine what structure exists on this category and how it reflects the structure of the Hopf algebra.

1 Categories, functors and natural transformations

Definition 1.1. A *category* $\mathcal{C}$ consists of

- a collection $\mathrm{Ob}(\mathcal{C})$ of so-called *objects*,
- for each pair of objects X and Y: a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of *morphisms* from X to Y, and
- for each triple of objects X, Y and Z: a map, called *composition*,

$$\begin{aligned} \circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) &\to \mathrm{Hom}_{\mathcal{C}}(X, Z), \\ (f, g) &\mapsto f \circ g, \end{aligned}$$

satisfying the following properties:

- (*Associativity*): $(f \circ g) \circ h = f \circ (g \circ h)$ for all $h \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$.
- (*Existence of unit*): For each object $X \in \mathrm{Ob}(\mathcal{C})$ there exists a morphism id_X such that $\mathrm{id}_X \circ g = g$ and $f \circ \mathrm{id}_X = f$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is called an *isomorphism*, if there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Observation 1.2. For any object $X \in \mathrm{Ob}(\mathcal{C})$ of a category $\mathcal{C}$, consider the set of endomorphisms of X:

$$\mathrm{End}_{\mathcal{C}}(X) := \mathrm{Hom}_{\mathcal{C}}(X, X)$$

The axioms of a category say that this is a monoid, with multiplication given by composition.

Examples 1.3.

1. The category $\mathrm{Vect}(k)$ of k-vector spaces:
   - Its objects $\mathrm{Ob}(\mathrm{Vect}(k))$ are all vector spaces over k.
   - Given vector spaces V and W, the set of morphisms $\mathrm{Hom}_{\mathrm{Vect}(k)}(V, W)$ is defined to be the set of k-linear maps from V to W.
   - Composition of two composable morphisms is defined to be the obvious composition of maps.

   We abbreviate $\mathrm{Hom}_k(\cdot, \cdot) := \mathrm{Hom}_{\mathrm{Vect}(k)}(\cdot, \cdot)$.

2. Let A be an algebra over k. The category $\mathrm{Mod}(A)$ has as objects all (left) A-modules and as morphisms A-linear maps. Composition is again naturally given. For the morphism sets we use the abbreviation $\mathrm{Hom}_A(\cdot, \cdot) := \mathrm{Hom}_{\mathrm{Mod}(A)}(\cdot, \cdot)$.

3. Given any category $\mathcal{C}$, there is a category $\mathcal{C} \times \mathcal{C}$, whose objects and morphisms are pairs of objects of $\mathcal{C}$ and pairs of morphisms of $\mathcal{C}$, respectively. Composition is then defined component-wise.

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Remark 1.4. In fact, the category $\mathrm{Vect}(k)$ of vector spaces (as well as the category of A-modules) has the special feature that its sets of morphisms naturally carry additional structure. They are themselves vector spaces over k. Furthermore, this extra structure on the category is compatible with the basic structure on the category: The composition of linear maps is a bilinear map. One says that the categories $\mathrm{Vect}(k)$ and $\mathrm{Mod}(A)$ are *k-linear* (or *enriched over the category* $\mathrm{Vect}(k)$).

Together with Observation 1.2 this leads to the following fact: For any vector space V, the set of endomorphisms $\mathrm{End}_k(V)$ is a monoid and also a vector space, such that the multiplication is bilinear. That is to say, $\mathrm{End}_k(V)$ is a k-algebra.

Definition 1.5. A *functor* $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to another category $\mathcal{D}$ consists of

- a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, and
- for each X and $Y \in \mathrm{Ob}(\mathcal{C})$, a map $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$,

together satisfying:

- $F(f \circ g) = F(f) \circ F(g) \quad$ for all $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$,
- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad$ for all $X \in \mathrm{Ob}(\mathcal{C})$.

Examples 1.6.

1. For any category $\mathcal{C}$ the identity on the class of objects and the identity on the morphisms combine to the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.
2. Given two composable functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, their composition is a functor $G \circ F : \mathcal{C} \to \mathcal{E}$.
3. The forgetful functor $F_A : \mathrm{Mod}(A) \to \mathrm{Vect}(k)$, which assigns to an A-module its underlying vector space and to an A-module morphism the same map as a k-linear map. This functor is *k-linear*, i.e. the maps $F_A : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_k(M, N)$ are k-linear.

Definition 1.7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ functors between them. A *natural transformation* $\eta : F \Rightarrow G$ from F to G is a family of maps $\left(\eta_X : F(X) \to G(X)\right)_{X \in \mathrm{Ob}(\mathcal{C})}$ indexed by $\mathrm{Ob}(\mathcal{C})$, such that for any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ in the category $\mathcal{C}$ the following square commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

η is called a natural *isomorphism* if η_X is an isomorphism for all $X \in \mathrm{Ob}(\mathcal{C})$.

Remarks 1.8.

1. Let $H : \mathcal{C} \to \mathcal{D}$ be a third functor. Two natural transformations $\eta : F \Rightarrow G$ and $\theta : G \Rightarrow H$ can be composed by composing their components:
$$(\theta \circ \eta)_X := \theta_X \circ \eta_X : F(X) \to H(X) \quad \text{for all } X \in \mathrm{Ob}(\mathcal{C}).$$
The natural endomorphisms $F \Rightarrow F$ of a functor thus form a monoid.
2. Now let $\mathcal{D} = \mathrm{Vect}(k)$. Then the natural transformations $F \Rightarrow G$ form a k-vector space, since their components are morphisms of k-vector spaces, who themselves form a vector space by Remark 1.4.
3. Combining 1. and 2.: The natural endomorphisms of a functor going into $\mathrm{Vect}(k)$ form a k-algebra.

Example 1.9. Recall the forgetful functor $F_A : \mathrm{Mod}(A) \to \mathrm{Vect}(k)$ for an algebra A, from Example 1.6.3. Let $a \in A$. We will give a natural endomorphism of the functor F_A. For every A-module M, acting with the element a defines a linear map

$$\rho_M(a) : M \to M, \quad x \mapsto a.x\ .$$

For any A-module morphism $f : M \to N$, the following holds by A-linearity:

$$f \circ \rho_M(a) = \rho_N(a) \circ f.$$

Thus, we obtain a natural endomorphism

$$\rho(a) := (\rho_M(a))_{M \in \mathrm{Ob}(\mathrm{Mod}(A))} : F_A \Rightarrow F_A$$

of the forgetful functor $F_A : \mathrm{Mod}(A) \to \mathrm{Vect}(k)$.

Note that, in general, $\rho_M(a)$ is not A-linear, that is, it is not a morphism in the category $\mathrm{Mod}(A)$, unless A is commutative. Hence, $\rho(a)$ is not a natural endomorphism of the identity functor on $\mathrm{Mod}(A)$.

2 Reconstruction of an algebra

Let A be an algebra. We have just seen that, using the forgetful functor $F_A : \mathrm{Mod}(A) \to \mathrm{Vect}(k)$, we can consider another algebra: the algebra $\mathrm{End}(F_A)$ of natural endomorphisms of F_A.

The question is how the algebras A and $\mathrm{End}(F_A)$ are related. We already have a map

$$\rho : A \to \mathrm{End}(F_A),$$

which maps $a \in A$ to the natural endomorphism $\rho(a)$ from Example 1.9. In fact, this is a homomorphism of algebras. Moreover, we have:

Proposition 2.1. *The map*

$$\begin{aligned} \rho : A &\to \mathrm{End}(F_A), \\ a &\mapsto (M \to M, x \mapsto a.x)_{M \in \mathrm{Ob}(\mathrm{Mod}(A))} \end{aligned}$$

is an isomorphism between the algebra A and the endomorphism algebra $\mathrm{End}(F_A)$ *of the forgetful functor* $F_A : \mathrm{Mod}(A) \to \mathrm{Vect}(k)$.

To summarize, we have thus fully reconstructed an algebra A from its category of representations and the forgetful functor to $\mathrm{Vect}(k)$. Note that the algebra $\mathrm{End}(F_A)$ is described purely in terms of the structures of $\mathrm{Mod}(A)$ and F_A as (k-linear) category and functor, respectively. To construct it we do not need $\mathrm{Mod}(A)$ to be the representation category of an algebra.

3 Monoidal categories and monoidal functors

A bialgebra is an algebra with additional structure: co-multiplication. This is mirrored by additional structure on the category of modules over a bialgebra. One can form tensor products of modules. We introduce the formal notion of a category with such additional monoidal structure:

Definition 3.1. A *monoidal category* (or *tensor category*) $(\mathcal{C}, \otimes, I, a, l, r)$ consists of

- a category $\mathcal{C}$,
- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called *tensor product*),
- an object $I \in \mathrm{Ob}(\mathcal{C})$ (called *unit*),
- a natural isomorphism $a : \otimes(\otimes \times \mathrm{id}) \Rightarrow \otimes(\mathrm{id} \times \otimes)$ (called *associativity constraint*), and
- natural isomorphisms $l : \otimes(I \times \mathrm{id}) \Rightarrow \mathrm{id}$ and $r : \otimes(\mathrm{id} \times I) \Rightarrow \mathrm{id}$ (called *left* and *right unit constraint*),

that together satisfy the following axioms (*Pentagon Axiom* and *Triangle Axiom*): The diagrams

$$\begin{tikzcd}
\big((U \otimes V) \otimes W\big) \otimes X \arrow[rr, "a_{U,V,W} \otimes \mathrm{id}_X"] \arrow[d, "a_{U \otimes V, W, X}"] & & \big(U \otimes (V \otimes W)\big) \otimes X \arrow[dd, "a_{U, V \otimes W, X}"'] \\
(U \otimes V) \otimes (W \otimes X) \arrow[d, "a_{U,V,W \otimes X}"] & & \\
U \otimes \big(V \otimes (W \otimes X)\big) & & U \otimes \big((V \otimes W) \otimes X\big) \arrow[ll, "\mathrm{id}_U \otimes a_{V,W,X}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(V \otimes I) \otimes W \arrow[dr, "r_V \otimes \mathrm{id}_W"] \arrow[dd, "a_{V,I,W}"'] & \\
& V \otimes W \\
V \otimes (I \otimes W) \arrow[ur, "\mathrm{id}_V \otimes l_W"'] &
\end{tikzcd}$$

commute for all objects $U, V, W, X \in \mathrm{Ob}(\mathcal{C})$.

Example 3.2.

1. The conventional tensor product of vector spaces and linear maps endows the category $\mathrm{Vect}(k)$ with the structure of a monoidal category. The tensor unit is the vector space given by the ground field k. Furthermore, for vector spaces U, V and W, we have canonical isomorphisms
$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W), \quad k \otimes V \cong V \quad \text{and} \quad V \otimes k \cong V.$$
2. Let H be a bialgebra. Then the category $\mathrm{Mod}(H)$ of (left) H-modules is a monoidal category as follows. Given two H-modules M and N we can endow their tensor product as vector spaces $M \otimes N$ with the structure of an H-module using the co-multiplication of H. Also, we can endow the vector space tensor unit k with an H-module structure using the co-unit of H. Finally, the same associativity, left unit and right unit constraints as for vector spaces also become isomorphisms of H-modules.

When one tries to reconstruct the bialgebra H from the monoidal category $\mathrm{Mod}(H)$ and the forgetful functor into $\mathrm{Vect}(k)$, one needs this functor to respect the additional monoidal structure:

Definition 3.3. Let $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ and $\mathcal{D} = (\mathcal{D}, \tilde{\otimes}, \tilde{I}, \tilde{a}, \tilde{l}, \tilde{r})$ monoidal categories. A *tensor functor* $(F, \varphi_0, \varphi_2)$ from $\mathcal{C}$ to $\mathcal{D}$ consists of

- a functor $F : \mathcal{C} \to \mathcal{D}$,
- an isomorphism $\varphi_0 : \tilde{I} \to F(I)$, and
- a natural isomorphism $\big(\varphi_2(U, V) : F(U) \tilde{\otimes} F(V) \to F(U \otimes V)\big)_{U,V \in \mathrm{Ob}(\mathcal{C})}$

such that the diagrams

$$\begin{tikzcd}
\big(F(U)\tilde{\otimes}F(V)\big)\tilde{\otimes}F(W) \arrow[r, "\tilde{a}_{F(U),F(V),F(W)}"] \arrow[d, "\varphi_2(U,V)\otimes \mathrm{id}_{F(W)}"] & F(U)\tilde{\otimes}\big(F(V)\tilde{\otimes}F(W)\big) \arrow[d, "\mathrm{id}_{F(U)}\otimes\varphi_2(V,W)"] \\
F(U\otimes V)\tilde{\otimes}F(W) \arrow[d, "\varphi_2(U\otimes V, W)"] & F(U)\tilde{\otimes}F(V\otimes W) \arrow[d, "\varphi_2(U, V\otimes W)"] \\
F\big((U\otimes V)\otimes W\big) \arrow[r, "F(a_{U,V,W})"] & F\big(U\otimes(V\otimes W)\big),
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{I}\tilde{\otimes}F(U) \arrow[r, "\tilde{l}_{F(U)}"] \arrow[d, "\varphi_0\tilde{\otimes}\mathrm{id}_{F(U)}"] & F(U) \\
F(I)\tilde{\otimes}F(U) \arrow[r, "\varphi_2(I,U)"] & F(I\otimes U) \arrow[u, "F(l_U)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(U)\tilde{\otimes}\tilde{I} \arrow[r, "\tilde{r}_{F(U)}"] \arrow[d, "\mathrm{id}_{F(U)}\tilde{\otimes}\varphi_0"] & F(U) \\
F(U)\tilde{\otimes}F(I) \arrow[r, "\varphi_2(U,I)"] & F(U\otimes I) \arrow[u, "F(r_U)"]
\end{tikzcd}$$

commute for all objects $U, V, W \in \mathrm{Ob}(\mathcal{C})$.

Example 3.4. Let H be a bialgebra. Then the forgetful functor $F_H : \mathrm{Mod}(H) \to \mathrm{Vect}(k)$ is a functor between monoidal categories. In fact, it becomes a tensor functor using $\varphi_0 = \mathrm{id}_k : k \to F_H(k) = k$ and $\varphi_2(M, N) = \mathrm{id}_{F_H(M \otimes N)} : F_H(M) \otimes F_H(N) \to F_H(M \otimes N)$ for H-modules M and N.

References

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