The quantum group $U_q(sl_2)$

Let the field we consider be $\mathbb{C}$.

q-numbers
We fix an invertible element $q \in \mathbb{C}$, $q \neq \pm 1$. So the fraction $\frac{1}{q-q^{-1}}$ is well-defined.
For any integer n define

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + ... + q^{-n+3} + q^{-n+1}.$$

If q is not a root of unity, then $[n] \neq 0$ for any non-zero integer n.
If q is a root of unity, then denote its order by d, i.e. $d \in \mathbb{N}$ is minimal such that $q^d = 1$. By the assumption $q \neq \pm 1$ we get $d > 2$.
Now define

$$e := \begin{cases} d, \text{ if } d \text{ is odd} \\ \frac{d}{2}, \text{ if } d \text{ is even.} \end{cases}$$

And set $d = e = \infty$ when q is not a root of unity.

Definition 1
Define $U_q(sl_2)$ to be the algebra generated by E, F, K and K^{-1}, such that

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F,$$
$$\text{and} \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Abbreviate $U_q := U_q(sl_2)$
$U_q(sl_2)$ is a quantum group, in fact the simplest one.

Proposition 2
The set $\{E^i F^j K^l\}_{i,j \in \mathbb{N}; l \in \mathbb{Z}}$ is a basis of $U_q(sl_2)$.

We would expect to recover the enveloping algebra of sl_2 from $U_q(sl_2)$ by setting $q = 1$, but this is impossible because of Definition 1. So we have to find another presentation for $U_q(sl_2)$.

Write $q = e^h$ and $K = q^H = e^{hH}$ and consider the limit $h \to 0$. Then the relations of Definition 1 imply (by differentiation at $h = 0$) that
$[H, E] = 2E, [H, F] = -2F$ and $[E, F] = H$. That is, we obtain the relations of $U(sl_2)$.

Proposition 3
The algebras $U_q(sl_2)$ and $U'_q(sl_2)$ are isomorphic, where $U'_q(sl_2)$ is generated by E, F, K, K^{-1} and L, which satisfy the following relations:

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = L,$$

$$(q - q^{-1})L = K - K^{-1}, \quad [L, E] = q(EK + K^{-1}E), \quad [L, F] = -q^{-1}(FK + K^{-1}F)$$

Remark that $U'_q(sl_2)$ is also defined for $q = 1$, which is $U_q(sl_2)$ not.

Proposition 4
If $q = 1$, then we have

$$U'_1(sl_2) \simeq U(sl_2)[K]/(K^2 - 1) \text{ and } U(sl_2) \simeq U'_1(sl_2)/(K - 1).$$

In particular the projection of $U'_1(sl_2)$ onto $U(sl_2)$ is given by

$$E \mapsto E, \quad F \mapsto F, \quad K \mapsto 1, \quad L \mapsto H.$$

Proposition 5
For q not a root of unity, the relations

$$\triangle(E) = 1 \otimes E + E \otimes K, \quad \triangle(F) = K^{-1} \otimes F + F \otimes 1,$$

$$\triangle(K) = K \otimes K, \quad \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K$$

endow $U_q(sl_2)$ with a Hopf algebra structure.

This Hopf algebra structure is neither commutative nor cocommutative.

Proposition 6
For any $u \in U_q(sl_2)$ the equation $S^2(u) = KuK^{-1}$ holds, i.e. the square of the antipode is inner.