LIE ALGEBRAS AND THEIR UNIVERSAL ENVELOPING ALGEBRAS

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Let $\mathbb{K}$ be an arbitrary field unless specified.

Lie Algebras

Definition 1. A **Lie algebra** $\mathcal{L}$ is a vector space over a field $\mathbb{K}$ with a bilinear map

$$[\cdot,\cdot] : \mathcal{L} \times \mathcal{L} \to \mathcal{L},$$

called the **Lie bracket**, satisfying the following conditions for all $x, y, z \in \mathcal{L}$:

(1) (Antisymmetry)

$$[x, x] = 0$$

(2) (Jacobi Identity)

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

Example 1. (1) If A is an associative algebra, we can endow it with the following Lie bracket $[x, y] := xy - yx \ \forall x, y \in A$. We denote the resulting Lie algebra by $\mathfrak{L}(A)$.

(2) $\mathfrak{gl}(n, \mathbb{K}) := M_n(\mathbb{K})$, the algebra of all $n \times n$ matrices with entries in $\mathbb{K}$, with the above defined bracket, forms a Lie algebra. Any subalgebra of this with the restriction of the same bracket will be a Lie algebra and is called *linear Lie algebra*.

(3) $\mathfrak{sl}(n, \mathbb{K}) := \{X \in \mathfrak{gl}(n, \mathbb{K}) | \operatorname{Tr}(X) = 0\}$ is an important linear Lie algebra. This vector space is generated by the following elements:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

They constitute the standard basis. The Lie bracket with respect to the standard basis is given by:

$$[H, E] = 2E \qquad [E, F] = H \qquad [H, F] = -2F.$$

(4) If $\mathcal{L}$ and $\mathcal{L}'$ are Lie algebras, we can equip the direct sum $\mathcal{L} \oplus \mathcal{L}'$ with a Lie bracket given by

$$[(x, x'), (y, y')] = ([x, y], [x', y'])$$

for all $x, y \in \mathcal{L}$ and $x', y' \in \mathcal{L}'$.

(5) Given a Lie algebra $\mathcal{L}$, we define the opposite Lie algebra $\mathcal{L}^{\mathrm{op}}$ as the vector space $\mathcal{L}$ with the Lie bracket $[-,-]^{\mathrm{op}}$ given by

$$[x, y]^{\mathrm{op}} = [y, x] = -[x, y].$$

Definition 2. A **Lie algebra homomorphism** is a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, such that

$$\varphi([X, Y]_{\mathfrak{g}}) = [\varphi(X), \varphi(Y)]_{\mathfrak{h}} \qquad \forall X, Y \in \mathfrak{g}.$$

It is called an isomorphism if it is bijective.

Example 2. (1) The canonical injections of Lie algebras into their direct sum and canonical projections of the direct sum onto the components are Lie algebra homomorphisms.
(2) $\mathrm{op} : \mathcal{L} \to \mathcal{L}^{\mathrm{op}}$ with $\mathrm{op}(x) := -x$ is a Lie algebra isomorphism.

Universal Enveloping Algebra

To any Lie algebra $\mathcal{L}$, we assign an associative algebra $U(\mathcal{L})$ called the *Universal Enveloping Algebra* of $\mathcal{L}$ and a homomorphism of Lie algebras $\iota_{\mathcal{L}} : \mathcal{L} \to \mathfrak{L}(U(\mathcal{L}))$.

Definition 3. Let $\mathcal{L}$ be a Lie algebra. Define $I(\mathcal{L})$ to be the two-sided ideal of the tensor algebra $T(\mathcal{L})$ generated by all elements of the form $xy - yx - [x, y]$, where x, y are elements of $\mathcal{L}$. We define the following quotient of $T(\mathcal{L})$ as the **universal enveloping algebra**;

$$U(\mathcal{L}) := T(\mathcal{L})/I(\mathcal{L}).$$

Remark 1. $U(\mathcal{L})$ is an associative algebra with unit.

Remark 2. We have a canonical linear map $\iota_{\mathcal{L}} : \mathcal{L} \to \mathfrak{L}(U(\mathcal{L}))$, given by the canonical injection of $\mathcal{L}$ into $T(\mathcal{L})$ and then projecting onto $U(\mathcal{L})$, which satisfies

$$\iota_{\mathcal{L}}[X, Y] = \iota_{\mathcal{L}}(X)\iota_{\mathcal{L}}(Y) - \iota_{\mathcal{L}}(Y)\iota_{\mathcal{L}}(X) \qquad \forall X, Y \in \mathcal{L}.$$

Hence, this is a Lie algebra homomorphism.

Theorem 1. ***(Universal Property)*** *Let $\mathcal{L}$ be a Lie algebra. Given any associative algebra A and any Lie algebra homomorphism $f : \mathcal{L} \to \mathfrak{L}(A)$, there exists a unique morphism of algebras $\varphi : U(\mathcal{L}) \to A$ such that $\varphi \circ \iota_{\mathcal{L}} = f$, i.e., such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{L} \arrow[rr, "\iota_{\mathcal{L}}"] \arrow[dr, "f"'] & & U(\mathcal{L}) \arrow[dl, dashed, "\exists!\varphi"] \\ & A & \end{tikzcd}$$

We can also express the above statement by the following natural bijection:

$$Hom_{Lie}(\mathcal{L}, \mathcal{L}(A)) \cong Hom_{Alg}(U(\mathcal{L}), A).$$

Corollary 1. (1) For any morphism of Lie algebras $f : \mathcal{L} \to \mathcal{L}'$, there exists a unique morphism of algebras $U(f) : U(\mathcal{L}) \to U(\mathcal{L}')$ such that $U(f) \circ \iota_{\mathcal{L}} = \iota_{\mathcal{L}'} \circ f$. We then have $U(\mathrm{id}_{\mathcal{L}}) = \mathrm{id}_{U(\mathcal{L})}$.
(2) If $f' : \mathcal{L}' \to \mathcal{L}''$ is another morphism of Lie algebras, then

$$U(f' \circ f) = U(f') \circ U(f).$$

(3) Let $\mathcal{L}$ and $\mathcal{L}'$ be Lie algebras and $\mathcal{L} \oplus \mathcal{L}'$ their direct sum. Then

$$U(\mathcal{L} \oplus \mathcal{L}') \cong U(\mathcal{L}) \otimes U(\mathcal{L}').$$

The main theorem about $U(\mathcal{L})$ gives a basis for $U(\mathcal{L})$ as a vector space.
Let $\{X_i\}_{i \in I}$ be a basis of $\mathcal{L}$. A set such as I always admits a simple ordering, i.e., a partial ordering in which every pair of elements is comparable.

Theorem 2. ***(Poincaré-Birkhoff-Witt Theorem)*** *Let $\{X_i\}_{i \in I}$ be a basis of $\mathcal{L}$ and suppose a simple ordering has been imposed on the index set I. Then the set of all monomials*

$$(\iota_{\mathcal{L}}(X_{i_1}))^{j_1} \dots (\iota_{\mathcal{L}}(X_{i_n}))^{j_n}$$

with $i_1 < \cdots < i_n$ and with all $j_k \geq 0$, is a basis of $U(\mathcal{L})$. In particular the canonical map $\iota_{\mathcal{L}} : \mathcal{L} \to U(\mathcal{L})$ is injective.

Remark 3. All universal enveloping algebras except $U(\{0\})$ are infinite dimensional.

$U(\mathcal{L})$ as a Hopf algebra

We are now in a position to put a Hopf algebra structure on $U(\mathcal{L})$.

- We define a **co-multiplication** Δ on $U(\mathcal{L})$ by $\Delta := \varphi \circ U(\delta)$, where δ is the diagonal map $x \mapsto (x, x)$ from $\mathcal{L}$ into $\mathcal{L} \oplus \mathcal{L}$ and φ is the isomorphism $U(\mathcal{L} \oplus \mathcal{L}') \to U(\mathcal{L}) \otimes U(\mathcal{L}')$.
- The **co-unit** is given by $\epsilon = U(0)$, where 0 is the zero morphism from $\mathcal{L}$ into the zero Lie algebra $\{0\}$.
- The **antipode** is given by $S = U(\mathrm{op})$, where op is the isomorphism from $\mathcal{L}$ to $\mathcal{L}^{\mathrm{op}}$.

Proposition 1. The universal enveloping algebra $U(\mathcal{L})$ is a cocommutative Hopf algebra for the map Δ as defined above.

Sketch of proof. The coassociativity axiom is satisfied as a consequence of the commutativity of the square

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "\delta"] \arrow[d, "\delta"] & \mathcal{L} \oplus \mathcal{L} \arrow[d, "\mathrm{id} \oplus \delta"] \\
\mathcal{L} \oplus \mathcal{L} \arrow[r, "\delta \oplus \mathrm{id}"] & \mathcal{L} \oplus \mathcal{L} \oplus \mathcal{L}.
\end{tikzcd}$$

The counit axiom is satisfied because of the commutativity of the diagram

$$\begin{tikzcd}
0 \oplus \mathcal{L} & \mathcal{L} \oplus \mathcal{L} \arrow[l, "0 \oplus \mathrm{id}"'] \arrow[r, "\mathrm{id} \oplus 0"] & \mathcal{L} \oplus 0 \\
& \mathcal{L} \arrow[ul, "\cong"] \arrow[u, "\delta"] \arrow[ur, "\cong"'] &
\end{tikzcd} .$$

The cocommutativity is ensured by the commutativity of the following triangle

$$\begin{tikzcd}
& \mathcal{L} \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\
\mathcal{L} \oplus \mathcal{L} \arrow[rr, "\tau"] & & \mathcal{L} \oplus \mathcal{L},
\end{tikzcd}$$

where $\tau : \mathcal{L} \oplus \mathcal{L} \to \mathcal{L} \oplus \mathcal{L}, (x, y) \mapsto (y, x)$ is the flip map.

The definition of S and Lemma 1 from seminar 5 imply that S is an antipode for $U(\mathcal{L})$. $\square$

Remark 4. All elements of $\mathcal{L}$ are primitive elements in $U(\mathcal{L})$.