1 Yang-Baxter equation and Artin Braid group

Definition 1.1. *A group F is called free if there exists $X \subset F$ such that $\langle X \rangle = F$ and for all groups H and maps $f : X \to H$ there exists a unique grouphomomorphism*

$$\phi : F \to H, \quad \phi|_X = f.$$

Remark 1.2.
- *The last property is the universal property of free groups.*
- *For an arbitrary set I one can construct a free group F with basis I.*

Proposition 1.3. *Let G be a group. Then there exists a free group F and a surjective grouphomomorphism $\phi : F \to G$.*

Remark 1.4. *Every group G is a quotient group for some free group F.*

$$G \simeq F/N$$

Let $R \subset F$ be a subset such that $\langle R \rangle = N$. The pair (X, R) is called the presentation of the group G:
$r = 1,\ r \in R$ are called relations.

Notation 1.5. $G = \langle X | R \rangle$

Example 1.6. *G cyclic $G = \langle x | x^n = 1 \rangle$*

Definition 1.7. *The Artin Braid group B_n is the group generated by $n - 1$ generaters $\sigma_1, .., \sigma_{n-1}$ and the following relations.*

1. *$\sigma_i \sigma_j = \sigma_j \sigma_i$ for all $i, j = 1, ...n - 1$ with $|i - j| \geq 2$ and*
2. *$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $i = 1, ..., n - 2$*

These relations are called „braid relations“.

Remark 1.8.
i) Note $|i - j| \geq 2$ in (1), so $\sigma_1 \sigma_2 \neq \sigma_2 \sigma_1$!

*ii) $B_n = \langle \sigma_1, ..., \sigma_{n-1} | \sigma_i^{-1} \sigma_j^{-1} \sigma_i \sigma_j = 1$ with $|i - j| \geq 2$,
$\sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1} \sigma_i = 1 \rangle$*

*iii) Obviously for a given $f \in Hom(B_n, G)$, G a group, the elements $s_i = f(\sigma_i),\ i = 1, ..., n - 1$ satisfy
1) $s_i s_j = s_j s_i\ |i - j| \geq 2$
2)$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$*

Lemma 1.9. *Let G be a group, $s_1, .., s_{n-1} \in G$ satisfying (1) and (2) (the braid relations), then there is a unique group homomorphism $f : B_n \longrightarrow G$ such that $s_i = f(\sigma_i)$ for $i = 1, ..n-1$.*

Corollary 1.10. *$B_2 \simeq \mathbb{Z}$ (Why?)*

Now we have a look at S_n, the symmetric group on n elements. A transposition $\tau \in S_n$ only permutates two elements. Let $s_1, ..., s_{n-1}$ be the transpositions, where s_i permutes i and $i+1$.
Observation:
$\{s_1, ..., s_{n-1}\}$ satisfy (1) and (2).
$S_n = \langle s_1, ..., s_{n-1} | s_i^{-1} s_j^{-1} s_i s_j = 1$ with $|i-j| \geq 2$, $s_{i+1}^{-1} s_i^{-1} s_{i+1}^{-1} s_i s_{i+1} s_i = 1$ for $i = 1, ..., n-2$, $s_i^2 = 1$ for $i = 1, ..., n-1 \rangle$

Lemma 1.11. *The group B_n $n \geq 3$ is nonabelian.*

From now on we analyse B_n from a geometric point of view.

Definition 1.12. *A geometric braid on $n \geq 1$ is a set $b \subset \mathbb{R}^2 \times I$ formed by n disjoint topological intervals (called strings), such that $\pi : \mathbb{R}^2 \times I \longrightarrow I$ maps each string homeomorphically onto I.*

$$b \cap (\mathbb{R}^2 \times \{0\}) = \{(1,0,0), (2,0,0), ..., (n,0,0)\}$$
$$b \cap (\mathbb{R}^2 \times \{1\}) = \{(1,0,1), ..., (n,0,1)\}$$

Definition 1.13. *Two geometric braids on n strings are isotopic if there is a continous map $F : b \times I \longrightarrow \mathbb{R}^2 \times I$ such that $F_s : b \longrightarrow \mathbb{R}^2 \times I$, sending $x \in b$ to $F(x,s)$ is a geometric braid on n strings and $F_0 = id_b : b \longrightarrow b$, $F_1(b) = b'$.*

The relation of isotopy is an equivalence relation.

- Can you explain why?
- Does the definition of isotopy is known to you from somewhere?

The equivalence classes are called braids on n strings.

Definition 1.14. *Let b_1, b_2 be two geometric braids. The product b_1b_2 is the set of points $(x, y, t) \in \mathbb{R}^2 \times I$, such that*

$$\begin{aligned} (x, y, 2t) \in b_1 \quad & 0 \le t \le \frac{1}{2} \\ (x, y, 2t-1) \in b_2 \quad & \frac{1}{2} \le t \le 1. \end{aligned}$$

b_1b_2 is a geometric braid on n strings.
Why?
Is there a neutral element of multiplication?

Notation 1.15. *$\mathcal{B}_n$ is the set of braids on n strings with multiplication.*

Lemma 1.16. *Each $B \in \mathcal{B}_n$ has a twosided inverse.*

Theorem 1.17. *There is a unique isomorphism*

$$\phi : B_n \longrightarrow \mathcal{B}_n$$

such that $\sigma_i \mapsto \sigma_i^+$.

Braid group representations from R-matrices

V vector space, c linear automorphism of $V \otimes V$. Define a linear automorphism c_i of $V^{\otimes n}$

$$c_i = \begin{cases} c \otimes id_{V^{\otimes(n-2)}}, & i = 1 \\ id_{V^{\otimes(i-1)}} \otimes c \otimes id_{V^{\otimes(n-i-1)}}, & 1 < i < n-1 \\ id_{V^{\otimes(n-2)}} \otimes c, & i = n-1 \end{cases}.$$

Lemma 1.18. *With c_i defined above we have $c_ic_{i+1}c_i = c_{i+1}c_ic_{i+1}$ for all i iff c is a solution of the Yang-Baxter equation.*

Corollary 1.19. *Let $c \in Aut(V \otimes V)$ be a solution of the Yang-Baxter equation. For any $n > 0$, there exists a group morphism*

$$\rho_n^c : B_n \longrightarrow Aut(V^{\otimes n})$$

such that $\rho_n^c(\sigma_i) = c_i$, $i = 1, ..., n-1$, ρ_n^c is unique.

Let R_λ be the universal R-matrix of Sweedler's 4-dim. Hopf algebra. Since $R_\lambda^2 \neq id$, this gives a representation with $c_i^2 \neq 1$. This is a representation that do not factor through S_n.