Bialgebras

1. Sweedler's sigma notation

Let (C, Δ, ϵ) be a coalgebra and x $\in$ C.

Then the element $\Delta(x) \in C \otimes C$ is of the form $\Delta(x) = \sum_i x_i' \otimes x_i''$. By omission of the subscrip we write instead $\Delta(x) = \sum_{(x)} x' \otimes x''$.

Using this notation we can rewrite the condition for coassociativity:

$\sum_{(x)}(\sum_{(x')}(x')' \otimes (x')'') \otimes x'' = \sum_{(x)} x' \otimes (\sum_{(x'')}(x'')' \otimes (x'')'')$.

By convention we write for both sides of the above equation

$\sum_{(x)} x' \otimes x'' \otimes x'''$ or $\sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)}$.

Applying the comultiplication to one of the components of the sum we get three equal expressions:

$\sum_{(x)} \Delta(x') \otimes x'' \otimes x'''$, $\sum_{(x)} x' \otimes \Delta(x'') \otimes x'''$, $\sum_{(x)} x' \otimes x'' \otimes \Delta(x''')$.

For these we write $\sum_{(x)} x' \otimes x'' \otimes x''' \otimes x''''$ or $\sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes x^{(4)}$.

More generally we inductively define maps $\Delta^{(n)} : C \to C^{\otimes(n+1)}$ for $n \geq 1$ by $\Delta^{(1)} = \Delta$ and

$\Delta^{(n)} = (\Delta \otimes id_{\otimes(n-1)}) \circ \Delta^{(n-1)} = (id_{C\otimes(n-1)} \otimes \Delta) \circ \Delta^{(n-1)}$.

By convention we write

$\Delta^{(n)}(x) = \sum_{(x)} x^{(1)} \otimes ... \otimes x^{(n+1)}$.

Using these conventions we can reformulate the condition for counitality as

$\sum_{(x)} \epsilon(x')x'' = x = \sum_{(x)} x'\epsilon(x'')$ for all $x \in C$

We get identities such as

$\sum_{(x)} x^{(1)} \otimes \epsilon(x^{(2)}) \otimes x^{(3)} \otimes x^{(4)} \otimes x^{(5)} = \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes x^{(4)}$,

by apllying the reformulation of the counitality condition to the left-hand side rewritten as

$\sum_{(x)} x^{(1)} \otimes (\epsilon \otimes id)(\Delta x^{(2)}) \otimes x^{(3)} \otimes x^{(4)}$.

We may further express the cocommutativity of the coalgebra C by

$\sum_{(x)} x' \otimes x'' = \sum_{(x)} x'' \otimes x'$ for all $x \in C$.

Also the relation $(f \otimes f) \circ \Delta = \Delta' \circ f$ for defining coalgebra morphisms can be reformulated as

$\sum_{(x)} f(x') \otimes f(x'') = \sum_{(f(x))} f(x)' \otimes f(x)''$.

2. Bialgebras

Let H be a vector space such that (H, μ, η) is an algebra and (H, Δ, ϵ) is a coalgebra. We have

Theorem 1: The following statements are equivalent.

a) μ and η are coalgebra morphisms.

b) Δ and ϵ are algebra morphisms.

Proof: We write down the commutative diagrams expressing that μ

$$\begin{tikzcd} H\otimes H \arrow[r, "\mu"] \arrow[d, "(id\otimes\tau\otimes id)\circ(\Delta\otimes\Delta)"'] & H \arrow[d, "\Delta"] \\ (H\otimes H)\otimes(H\otimes H) \arrow[r, "\mu\otimes\mu"] & H\otimes H \end{tikzcd} \qquad \begin{tikzcd} H\otimes H \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "\mu"'] & K\otimes K \arrow[d, "id"] \\ H \arrow[r, "\epsilon"] & K \end{tikzcd}$$

and η are coalgebra morphisms.

$$\begin{tikzcd} K \arrow[r, "\eta"] \arrow[d, "id"'] & H \arrow[d, "\Delta"] \\ K\otimes K \arrow[r, "\eta\otimes\eta"] & H\otimes H \end{tikzcd} \qquad \begin{tikzcd} K \arrow[rr, "\eta"] \arrow[dr, "id"'] & & H \arrow[dl, "\epsilon"] \\ & K & \end{tikzcd}$$

Now it is easy to see that these are the same as the ones expressing that Δ and ϵ are algebra morphisms:

$$\begin{tikzcd} H\otimes H \arrow[r, "\mu"] \arrow[d, "(id\otimes\tau\otimes id)\circ(\Delta\otimes\Delta)"'] & H \arrow[d, "\Delta"] \\ (H\otimes H)\otimes(H\otimes H) \arrow[r, "\mu\otimes\mu"] & H\otimes H \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "\eta"] \arrow[d, "id"'] & H \arrow[d, "\Delta"] \\ K\otimes K \arrow[r, "\eta\otimes\eta"] & H\otimes H \end{tikzcd}$$

$$\begin{tikzcd} H\otimes H \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "\mu"'] & K\otimes K \arrow[d, "id"] \\ H \arrow[r, "\epsilon"] & K \end{tikzcd} \qquad \begin{tikzcd} K \arrow[rr, "\eta"] \arrow[dr, "id"'] & & H \arrow[dl, "\epsilon"] \\ & K & \end{tikzcd}$$

$\square$

Definition 1: A bialgebra is a quintuple $(H, \mu, \eta, \Delta, \epsilon)$ such that (H, μ, η) is an algebra, (H, Δ, ϵ) is a coalgebra and one of the equivalent conditions of Theorem 1 is true.

Using Sweedler's sigma notation we can rewrite the condition $\Delta(xy) = \Delta(x)\Delta(y)$ as follows:

$\sum_{(xy)}(xy)' \otimes (xy)'' = \sum_{(x)(y)} x'y' \otimes x''y''$

We also get $\Delta(1) = 1 \otimes 1$, $\epsilon(xy) = \epsilon(x)\epsilon(y)$, $\epsilon(1) = 1$.

We now introduce the opposite coalgebra.

Let (C, Δ, ϵ) be a coalgebra. Comsider the function $\Delta^{op} = \tau_{C,C} \circ \Delta$ where $\tau_{C,C}$ denotes the flip $\tau_{C,C} : C \otimes C \to C \otimes C : c_1 \otimes c_2 \mapsto c_2 \otimes c_1$.

Then $C^{cop} := (C, \Delta^{op}, \epsilon)$ is a coalgebra which we call the opposite coalgebra.

Similarly, if (A, μ, η) is an Algebra then (A, μ^{op}, η) is an algebra which we call the opposite algebra and denote by A^{op}. This gives us the following result:

Let $H = (H, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. Then $H^{op} = (H, \mu^{op}, \eta, \Delta, \epsilon)$, $H^{cop} = (H, \mu, \eta, \Delta^{op}, \epsilon)$ and $H^{op\ cop} = (H, \mu^{op}, \eta, \Delta^{op}, \epsilon)$ are also bialgebras.

Theorem 2: The dual of a finite dimensional bialgebra is again a bialgebra.

Proof: We know that the dual of any coalgebra is a coalgebra and that of any finite dimensional algebra is an algebra. All we need to do is show that the conditions of Theorem 1 are true. $\square$

Examples:

Let $(G, *)$ be a group, $C = K[G] := \oplus_{g \in G} Kg$ be the vector space with basis G.

The group multiplication and its neutral element naturally make C an algebra.

We define a coalgebra structure on C via $\Delta(x) = x \otimes x,\ \epsilon(x) = 1$.

Then we have

$\Delta(xy) = xy \otimes xy = (x \otimes x)(y \otimes y) = \Delta(x)\Delta(y)$ and $\epsilon(xy) = 1 = \epsilon(x)\epsilon(y)$.

This shows that Δ and ϵ are algebra morphisms which makes $K[G]$ a bialgebra.

The dual algebra $C^* = K[G]^*$ is the algebra of functions on G with values in K.

In case G is finite the dual of the finite dimensional algebra $K[G]$ has a coalgebra structure and therefore $K[G]^*$ again is a bialgebra.

Comultiplication and counit are given by

$\Delta(f)(x \otimes y) = f(xy)$ and $\epsilon(f) = f(e)$.

Theorem 3: Let K be a field, $n \geq 2$. There is no bialgebra structure on $M_n(K)$ such that the underlying algebra structure is that of the matrix algebra.

Proof: Suppose we had a bialgebra structure on $M_n(K)$, then the counit $\epsilon : M_n(K) \to K$ is an algebra morphism. The kernel of ϵ is a two - sided ideal of $M_n(K)$, so it has to be either 0 or all of $M_n(K)$. Since $\epsilon(1) = 1$ we have $ker(\epsilon) = 0$ and obtain a contradiction to $dim(M_n(K)) > dim(K)$. $\square$

The tensor bialgebra

Let M be a K - vector space. Consider the tensor algebra (T(M), i).

We can define a coalgebra structure on T(M):

Let α, β be elements of T(M). By convention we write $\alpha\bar{\otimes}\beta \in T(M) \otimes T(M)$.

Consider the linear function $f : M \to T(M) \otimes T(M) : m \mapsto m\bar{\otimes}1 + 1\bar{\otimes}m$.

By application of the universal property of the tensor algebra we get an algebra morphism

$\Delta : T(M) \to T(M) \otimes T(M)$ such that $\Delta i = f$.

Again for $g : M \to TM \otimes TM \otimes TM : m \mapsto m\bar{\otimes}1\bar{\otimes}1 + 1\bar{\otimes}m\bar{\otimes}1 + 1\bar{\otimes}1\bar{\otimes}m$ the same property ensures the existance of an unique map $\bar{g}$ such that the following diagram commutes:

$M \ - \ i \longrightarrow TM$

$f \downarrow \qquad\qquad \swarrow \bar{g}$

$TM \otimes TM \otimes TM$

Since we have

$(\Delta \otimes I)\Delta(m) = (\Delta \otimes I)(m\bar{\otimes}1 + 1\bar{\otimes}m) = m\bar{\otimes}1\bar{\otimes}1 + 1\bar{\otimes}m\bar{\otimes}1 + 1\bar{\otimes}1\bar{\otimes}m = g(m)$

and $(I \otimes \Delta)\Delta(m) = (I \otimes \Delta)(m\bar{\otimes}1 + 1\bar{\otimes}m) = m\bar{\otimes}1\bar{\otimes}1 + 1\bar{\otimes}m\bar{\otimes}1 + 1\bar{\otimes}1\bar{\otimes}m = g(m)$

We therefore have $(\Delta \otimes I)\Delta(m) = (I \otimes \Delta)\Delta(m)$ which proves that Δ is coassociative.

For a counit we apply the universal property to the function $0 : M \to K$ and receive an algebra morphism $\epsilon : T(M) \to K$ with $\epsilon(m) = 0 \ \ \forall m \in i(M)$.

The same universality argument as above shows that ϵ is a counit.

This makes T(M) a bialgebra.