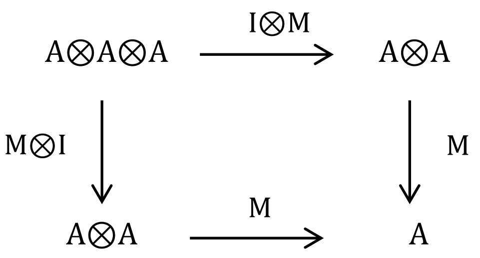
University Hamburg 28.10.2016
Seminar on Hopf algebras Matthias Lienau

Coalgebras

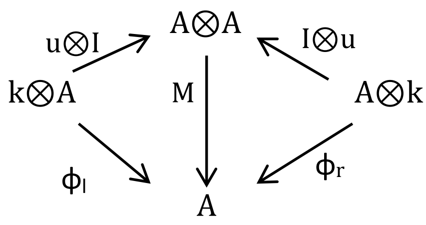
Let k be a field and each map a linear map.

Definition 1

We define an algebra as a triple (A,M,u) with A a vector space over k, $M: A \otimes A \to A$ and $u: k \to A$ maps such that the following diagrams commute:



Associativity



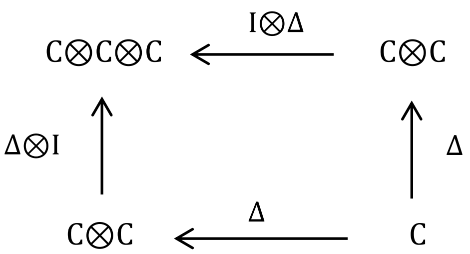
Unitary property

With the natural isomorphisms $\phi_r(a\otimes x)=ax$ and $\phi_l(x\otimes a)=xa$.

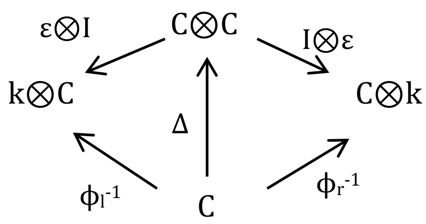
If we say A is an algebra, we mean a triple (A,M_A,u_A).

Definition 2

We define a coalgebra as a triple (C, Δ, ε) with C a vector space over k, $\Delta: C \to C \otimes C$ and $\varepsilon: C \to k$ maps such that the following diagrams commute:



Coassociativity



Counitary property

With the natural isomorphisms $\phi_l^{-1}(c) = 1 \otimes c$ and $\phi_r^{-1}(c) = c \otimes 1$.

If we say C is a coalgebra, we mean a triple $(C,\Delta_C,\varepsilon_C)$.

Examples 3

3.1) The field k is a coalgebra with $\Delta(x) = 1 \otimes x$ for any $x \epsilon k$ and $\varepsilon = id$.

3.2) Let S be a non-empty set and kS a vector space with basis S. Then kS is a coalgebra with $\Delta(s) = s \otimes s$ and $\varepsilon(s) = 1$ for any $s \epsilon S$ and called the coalgebra of a set.

3.3) For $n \in \mathbb{N}$ let $M(n,k)$ be a vector space over k with dimension n^2 and basis $(e_{ij})_{i,j\in\{1,\dots,n\}}$, e.g. the quadratic matrices of size n. This is a coalgebra with $\Delta(e_{ij})=\sum_k e_{ik}\otimes e_{kj}$ and $\varepsilon(e_{ij})=\delta_{ij}$. It's called the matrix coalgebra.

3.4) Let G be a finite group and $k(G)$ the vector space $\{f: G \to k\}$. Using the fact that $\rho': k(G) \otimes k(G) \to k(GxG),\ \ f \otimes g \mapsto ((x,y) \mapsto f(x)g(y))$ is an isomorphism, we can define: $\rho' \circ \Delta(f)(x,y) = f(xy)$ and $\varepsilon(f) = f(e)$. Now $k(G)$ is a coalgebra.

3.5) Sweedler's 4-dimensional Hopf algebra
Consider a field k with char(k)≠2. Let H be the algebra given by generators and relations as follows. H is generated as a k-algebra by c and x satisfying the relations
$c^2=1,\ x^2=0,\ xc=-cx.$
H also becomes a coalgebra with:
$\Delta(1) = 1 \otimes 1,\quad \Delta(c) = c \otimes c,\quad \Delta(x) = 1 \otimes x + x \otimes c,\quad \Delta(cx) = c \otimes cx + cx \otimes 1$
$\varepsilon(1) = 1,\quad \varepsilon(x) = 0,\quad \varepsilon(c) = 1,\quad \varepsilon(cx) = 0.$

If V is a vector space over k and we have $f \in V^*, v \in V$ we will write $< f, v >$ instead of $f(v)$.

Proposition 4
Let V and W be k-vector spaces. The map $\rho: V^* \otimes W^* \to (V \otimes W)^*$ given by

$< \rho(f \otimes g), v \otimes w > := < f, v >< g, w >$ for $f \in V^*, g \in W^*, v \in V, w \in W$ is injective.

Proposition 5
Let (C, Δ, ε) be a coalgebra. We receive an algebra (C^*, M, u) by defining:

$M = \Delta^* \circ \rho: C^* \otimes C^* \to C^*$
$u = \varepsilon^* \circ \varphi: k \to C^*$

with $\varphi: k \to k^*, < \varphi(a), x > = ax$

Proposition 6
Let (A, M, u) be a finite-dimensional algebra. We receive a coalgebra $(A^*, \Delta, \varepsilon)$ by defining:

$\Delta := \rho^{-1} \circ M^*: A^* \to A^* \otimes A^*$
$\epsilon := \varphi^{-1} \circ u^*: A^* \to k$

with $\varphi^{-1}: k^* \to k,\ \varphi^{-1}(f) = < f, 1 >$

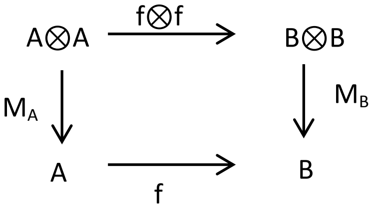
Proposition 7
For two coalgebras C,D we get a new coalgebra C⊗D if we define the following:

$\Delta_{C\otimes D} = (I \otimes T \otimes I) \circ (\Delta_C \otimes \Delta_D): C \otimes D \to C \otimes D \otimes C \otimes D$
$\varepsilon_{C\otimes D} = \phi_r \circ (\epsilon_C \otimes \epsilon_D): C \otimes D \to k$

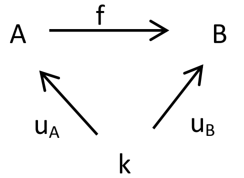
where T is the "twist" $T: C \otimes D \to D \otimes C,\ T(c \otimes d) = d \otimes c)$

<u>Definition 8</u>

Let A,B be algebras and $f: A \to B$ a map. We call f an algebra homomorphism if the following diagrams commute:



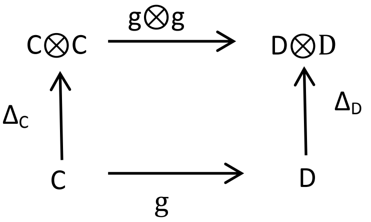
multiplicative

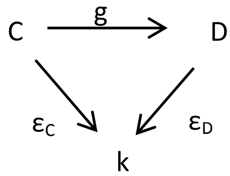


unit preserving

<u>Definition 9</u>

Let C,D be coalgebras and $g: C \to D$ a map. We call g a coalgebra homomorphism if the following diagrams commute:





<u>Remark 10</u>

This concept works very well with proposition 4 and 5 which means:

If $f: A \to B$ is an algebra homomorphism, $f^*: B^* \to A^*$ is a coalgebra homomorphism (this only makes sense for A being finite-dimensional) and if $g: C \to D$ is a coalgebra homomorphism, $g^*: D^* \to C^*$ is an algebra homomorphism.

<u>Definition 11</u>

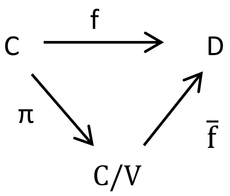
Let C be a coalgebra and V a subspace of C. We call V a (two-sided) coideal if:

1.) $\Delta(V) \subseteq V \otimes C + C \otimes V$
2.) $\varepsilon(V)=\{0\}$

<u>Theorem 12</u> (fundamental homomorphism theorem for coalgebras)

Let C be a coalgebra, V a coideal, $\pi: C \to C/V$ the natural projection and $f: C \to D$ a coalgebra map. Then

a.) C/V has a unique coalgebra structure such that π is a coalgebra map.
b.) ker f is a coideal.
c.) If $V \subseteq \ker f$ there is a unique coalgebra map $\bar{f}$ such that



commutes.