I. Dynamical Systems:

An introduction (resp. review) of basic concepts using Lotka-Volterra type equations in population dynamics

1. Population growth of single species

1.1. Exponential growth (unlimited resources)

- $x_{n+1} = R\, x_n$ discrete generations 0, 1, 2, 3, ...

  R — growth rate, intrinsic, constant

  $\Rightarrow \quad x_n = R\, x_{n-1} = \dots = R^n x_0$ polyn. growth

  explodes, if $R > 1$.

- $\dfrac{x(t+\Delta t) - x(t)}{\Delta t}$

  $x = x(t) = \#$ population at time $t \in \mathbb{N}$

  not differentiable, but if $x(t)$ is large enough, ± 1 looks negligibly small $\longrightarrow$

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} =: \dot{x}$$

$\dot{x} = r\, x$ population in continuous time $t \in \mathbb{R}$

r — growth rate per capita

$\Rightarrow \quad x(t) = x(0)\, e^{rt}$

exponential growth,

explodes, if $r > 0$.

1.2. Logistic growth (limited resources)

$$\dot{x} = r x \left(1 - \frac{x}{K}\right), \quad K > 0 \text{ — carrying capacity (space, food etc.) of environment}$$

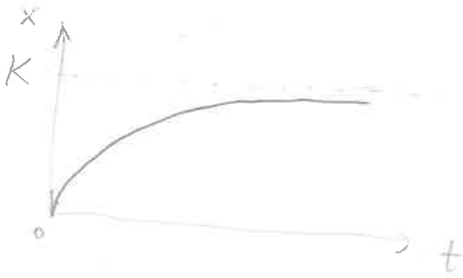
$$\begin{cases} x \nearrow & \text{if } 0 < x < K \quad \text{"abundance"} \\ x \searrow & \text{if } x > K \quad \text{"shortage"} \end{cases}$$

- explicit solutions (approach of differential equations)

$$\frac{dt}{dx} = \frac{1}{r}\left(\frac{1}{x} + \frac{1}{K-x}\right) \Rightarrow x(t) = \frac{K x(0) e^{rt}}{K + x(0)(e^{rt} - 1)}$$

for small $t > 0$, $x(t) \approx \dfrac{K x(0) e^{rt}}{K} = x(0) e^{rt}$ exponential growth

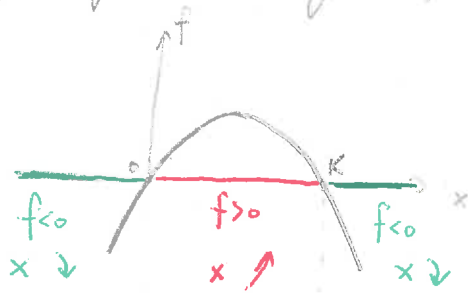
for $t \to \infty$, $x(t) \to \dfrac{K x(0) e^{rt}}{x(0) e^{rt}} = K$



"logistic function": exponentially increasing for small t, asymptotically growing to K.

- qualitative behavior (approach of dynamical systems)

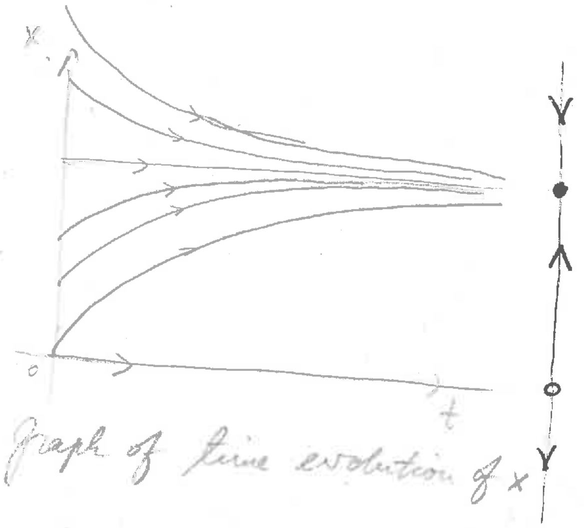
$$f(x) := r x \left(1 - \frac{x}{K}\right) \begin{cases} < 0 & \text{—} \\ = 0 & x = 0, K \\ > 0 & \text{—} \end{cases}$$



$$\dot{x} = f(x) \Rightarrow \begin{cases} x \text{ increases if } f > 0 \\ x \text{ decreases if } f < 0 \end{cases}$$

An equilibrium of $\dot{x} = f(x)$ is x s.t. $f(x) = 0$.

phase portrait



Graph of time evolution of x

1.3 The recurrence relation $x_{n+1} = R x_n (1-x_n)$, $n \geq 0$

A discrete model under limited resources:

$$X_{n+1} = R x_n \left(1 - \frac{x_n}{K}\right)$$

Note that $X_{n+1} \geq 0 \Rightarrow x_n \leq K$

$$\left. x_n = R x_{n-1}\left(1 - \frac{x_{n-1}}{K}\right) \Rightarrow x_n^{max} = R \cdot \frac{K}{2}\left(1 - \frac{\frac{K}{2}}{K}\right) = R \cdot \frac{K}{4} \right\}$$

$$\Rightarrow R \leq 4$$

Aim: describe $\{x_n\}_{n\geq 0}$ especially for n large (i.e. asymptotic behavior of x_n) for $0 < R \leq 4$

WLOG, replace x_n with $K x_n$, then

$$X_{n+1} = R x_n (1 - x_n), \quad n \geq 0$$

1.4. Stable and unstable fixed points

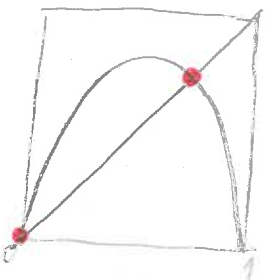
$$x \longmapsto F(x) := R\,x\,(1-x) \qquad \text{for} \quad x \in [0,1]$$

defines a discrete dynamical system on $[0,1]$ by:
(Since we assume $0 < R \leq 4$, $F([0,1]) \subseteq [0,1]$)

$$x_0,\ x_1 = F(x_0),\ x_2 = F(x_1) = F^{(2)}(x_0), \ldots,\ x_{n+1} = F(x_n) = F^{(n+1)}(x_0), \ldots,$$

for discrete time set $\{0,1,2,\ldots\} = \mathbb{N} \cup \{0\}$.

A <u>fixed point</u> of the dynam. system $x_{n+1} = F(x_n)$, is x such that $F(x) = x$, i.e. intersection of the graph $y = F(x)$ and the graph $y = x$ on x-y plane.



A fixed point $x = p$ is called <u>stable</u>, if nearby points remain nearby for all time, i.e.
$\forall$ nbhd U of p $\exists$ nbhd U_1 of p s.t.

$$\forall x_0 \in U_1, \quad \{x_n\}_{n \geq 0} \subseteq U$$

⌈ conti: $\forall x_0 \in U_1$, $x(t) \in U$ $\forall_{t \geq 0}$ ⌋

It is <u>asymptotically stable</u>, if U_1 can be chosen s.t. $x_n \to p$ as $n \to \infty$

We describe the stability of fixed points in relation with the parameter R

- $0 < R \leq 1$

$$F(x) - x = Rx(1-x) - x \leq x(1-x) - x = -x^2 < 0 \quad \forall x \in (0,1]$$

$\Rightarrow \forall\, x_0 \in (0,1]$, we have

$$x_0 > x_1 > x_2 > \dots$$

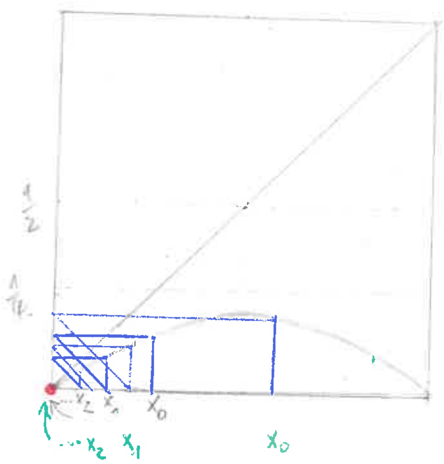
$\Rightarrow \{x_n\}$ is monot. decreasing, bounded from below by 0,

$\bar{x} := \lim_{n\to\infty} x_n$ exists, but $F(\bar{x}) = F(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} F(x_n)$

$$= \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = \bar{x}$$

So being a fixed point, $\bar{x}$ must equal ~~to~~ 0.

$\Rightarrow \quad x_n \xrightarrow[n\to\infty]{} 0 \quad \Rightarrow 0$ is asymp. stable.



$$R \leq 1 \Rightarrow F_{max} = \frac{R}{4} \leq \frac{1}{4}.$$

- $R > 1$

additional f.p. $\quad F(x) = x \;\Rightarrow\; x = 0$ or $x = \frac{R-1}{R} =: p$

Stability of p ?

$$F(x) - p = F(x) - F(p) = \frac{dF}{dx}(c)\,(x-p) \quad \text{for some } c \text{ between } x, p.$$

For x close to p, such c is close to p, so we can use

$$\frac{dF}{dx}(c) \approx \frac{dF}{dx}(p) = R(1-2x)\Big|_{x=\frac{R-1}{R}} = R\left(1-2\frac{R-1}{R}\right) = 2-R$$

$\Rightarrow$ if $|2-R| < 1$, then $\left|\frac{dF}{dx}(p)\right| < 1 \Rightarrow \left|\frac{dF}{dx}(c)\right| < 1 \Rightarrow |F(x) - p| < |x-p|$

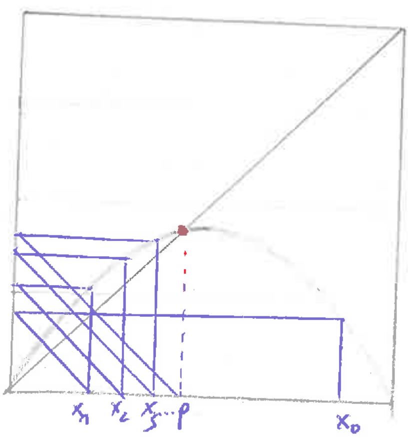
i.e. $1 < R < 3$

i.e. x_n approaches to p with smaller and smaller distance, as $n \to \infty$

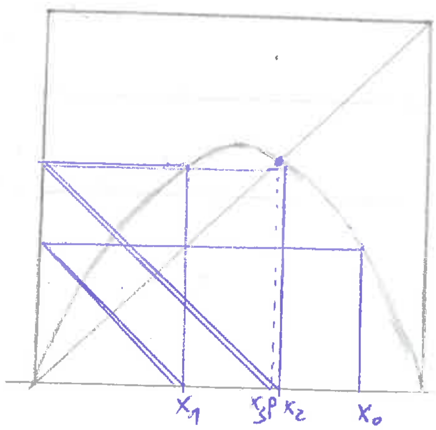
p is the unique f.p. in its nbhd $\Longrightarrow$ $x_n \to p$ as $n \to \infty$.

i.e. p is asymp. stable

Ex. Show that p is a global attractor, i.e. all orbits in $(0,1)$ converge to p.



$1 < R \leq 2$ ultimately monot. increasing

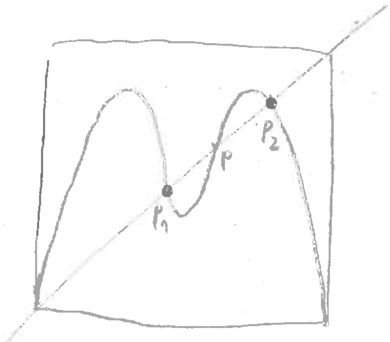


$2 < R < 3$, ultimately alternating.

1.3. Bifurcations

- $3 \leq R \leq 4$

For $3 < R \leq 4$, F possess periodic orbits.

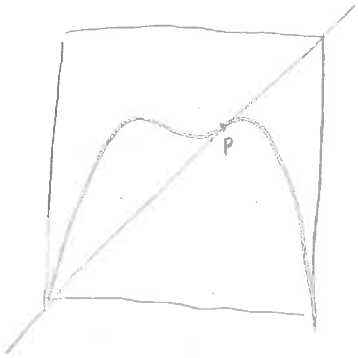


$R = 3.4$.

Graph of $F^{(2)}$

$F(p_1) = p_2$, $F(p_2) = p_1$

$\{p_1, p_2\}$ is a periodic orbit of period 2 for F.



$R = 2.7$

Indeed, for $1 < R < 3$, p is global attractor on $(0,1)$, so F can not have per. orbits in this case.

As R becomes larger, there are more and more per. orbits of F, they are firstly all of form 2^n,

for $R = 3.5700$, $\exists$ infinitely many per. orbits of F

for $R = 3.6786$, per. orbits of odd period appear.

for $R = 3.82\ldots$ all periods occur!

$\longrightarrow$ the system behaves chaotically.

$\boxed{R=3}$ is a bifurcation value of R

$R<3$ (slightly smaller), p is (asymp.) stable, no oscillations,

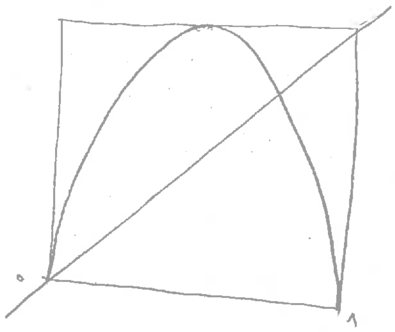
$R>3$ (slightly larger), p is unstable, oscillations appear,

i.e. the system behaves drastically different, as R crosses 3.

1.6. Chaotic motion

A sense of chaos : consider $\boxed{R=4}$.

(many other values of $R \in (3,4)$ behave similarly)



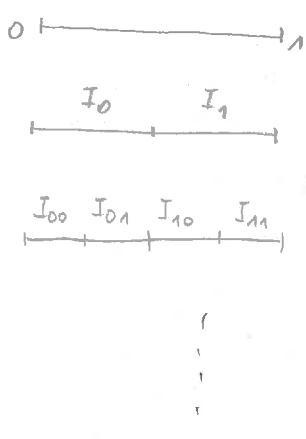
$F(x) = 4x(1-x)$

$F: [0,1] \to [0,1]$ is a "doubling" map

$I_0 := [0, \frac{1}{2}] \xrightarrow{\cong} [0,1]$

$I_1 := [\frac{1}{2}, 1] \xrightarrow{\cong} [0,1]$

bijectively



$I_0, I_1 \xrightarrow[F]{\cong} [0,1]$ rank 1 intervals

$I_{00}, I_{01}, I_{10}, I_{11} \xrightarrow[F^{(2)}]{\cong} [0,1]$ rank 2 intervals

Binary expansion of $x \in [0,1)$ $\longleftrightarrow$ $x \in I_0,\ F(x) \in I_1,\ F^{(2)}(x) \in I_0,\ F^{(3)}(x) \in I_0$

$x \in I_{0100\ldots}$

Unpredictability

sensitivity of initial value / measurement

two very close points

$x \in I_{0100101010100\ldots101\ldots}$

$x' \in I_{0100101010100\ldots110\ldots}$

$\langle$ same $\rangle$

mean that up from an n step they evolve quite unrelatedly

Often discrete models offer quite unregular motions than their conts. conterparts.

2 Population for 2 species (predator - prey)

2.1 Lotka-Volterra equations for predator-prey systems

austrian (1925), italian (1926)

End of World War I : more predatory fish in Adriatic Sea. why?

$$\begin{cases} \dot{x} = x(a - by) \\ \dot{y} = y(-c + dx) \end{cases}$$

x — prey < a – intrinsic growth, b – predator effect

y — predator < c – intrinsic, d – prey effect.

$a, b, c, d > 0$.

2.2. Solutions of diff. eqs.

(*) $\dot{x} = \frac{dx}{dt} = f(t, x)$, $x = (x_1, x_2, \dots, x_n)$ unknown

or componentwise $f: U \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given, usually C^1

$$\dot{x}_i = f_i(t, x_1, x_2, \dots, x_n), \quad i = 1, 2, \dots, n$$

A solution of the ODE (*) is a map

$$x: t \mapsto x(t)$$

defined on some interval $I \subseteq \mathbb{R}$ into $\mathbb{R}^n$ such that

$$\frac{d}{dt} x_i(t) = f_i(t, x_1(t), x_2(t), \dots, x_n(t)) \quad \text{for all } i = 1, \dots, n, \ t \in I$$

Picture :

$f(t, x(t))$ gives direction/velocity for x at time t

a solution gives a path ...

Theorem 1 (Existence) Consider $\dot{x} = f(t,x)$ where f is conts. on $R = \{(t,x) \; t_0-\delta < t < t_0+\delta, \; x_0 - \varepsilon < x < x_0+\varepsilon\}$ around (t_0, x_0). Then there exists $\delta_1 > 0$ s.t. $x = x(t)$ is a solution of $\dot{x} = f(t,x)$ for $t_0 - \delta_1 < t < t_0 + \delta_1$ with $x(t_0) = x_0$

Theorem 2 (Uniqueness) If additionally to Theorem 1, $\frac{\partial f}{\partial x}$ is conts. on R. Then there exists $\delta_2 > 0$ s.t. $x = x(t)$ is a unique solution for $t_0 - \delta_2 < t < t_0 + \delta_2$ with $x(t_0) = x_0$

Throughout, we always deal with C^1 maps f.

Thm 1-2 are "local" theorems

$\dot{x} = 1 + x^2 \quad \Rightarrow \quad x = x(t) = \tan(t + c)$

explodes in finite time

Theorem 3 (Global for cpt mnf) The eqn. $\dot{x} = f(t,x)$ where f is C^1 on a cpt. mnf. $M \ni x$, has a solution defined for all $t \in \mathbb{R}$.

We deal with <u>autonomous</u> systems $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ throughout, i.e. f defines a vector field on $\mathbb{R}^n$ which does not change as time t passes.

Three types of solutions can occur:

- (i) Equilibria: $x(t) \equiv x$ for all t, i.e. $f(x) = 0$, zeros of f.

(ii) Periodic orbits: $x(0) = x(T)$ for some $T > 0$ but $x(0) \neq x(t)$ $\forall_{0<t<T}$

T is the (minimal) <u>period</u> of x, x is called <u>a per. point</u>, its orbit $\{x(t) : t \in \mathbb{R}\} = \{x(t) : 0 \leq t \leq T\}$ is called a <u>per. orbit</u>.

(iii) $t \mapsto x(t)$ injective, the orbit looks topo. like a line

2.3. <u>Analysis of Lotka-Volterra predator-prey eqn.</u>

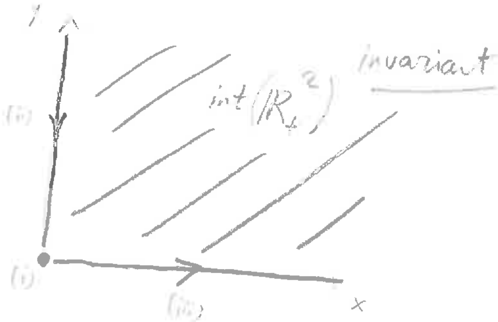
$$\begin{cases} \dot{x} = x(a - by) \\ \dot{y} = y(-c + dx) \end{cases}$$

has 3 trivial solutions:

(i) $x(t) = y(t) \equiv 0$

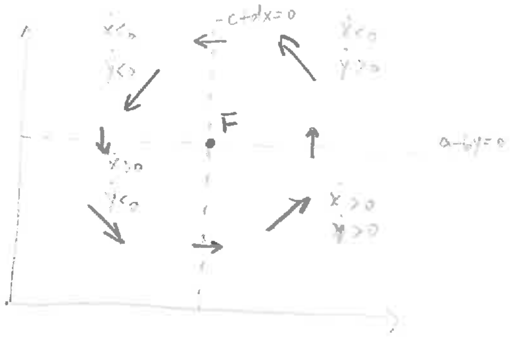
(ii) $x(t) \equiv 0$, $y(t) = y(0)e^{-ct}$ $\forall\, y(0) > 0$

(iii) $y(t) \equiv 0$, $x(t) = x(0)e^{at}$ $\forall_{x(0) > 0}$.



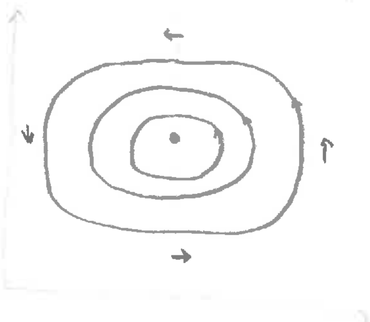
To find other equilibria, consider the isoclines:

$$\begin{cases} x(a-by)=0 \\ y(-c+dx)=0 \end{cases} \xrightarrow[y\neq 0]{x\neq 0} \bar{x}=\frac{c}{d},\quad \bar{y}=\frac{a}{b}$$



$\exists$ per. orbits around F ?

We show every point $\in \mathrm{int}(\mathbb{R}^2_+)$ except F follows a per. orbit.



using Lyapunov function.

$$\begin{cases} \dfrac{\dot{x}}{x} = a - by \\[2ex] \dfrac{\dot{y}}{y} = -c + dx \end{cases} \Rightarrow \frac{\dot{x}}{x}(c-dx)+\frac{\dot{y}}{y}(a-by)=0 \Rightarrow \dot{x}\left(\frac{c}{x}-d\right)+\dot{y}\left(\frac{a}{y}-b\right)=0$$

$$\Rightarrow \frac{d}{dt}\Big[\underbrace{(c\log x - dx)}_{\substack{\| \\ d(\frac{c}{d}\log x - x)}} + \underbrace{(a\log y - by)}_{\substack{\| \\ b(\frac{a}{b}\log y - y)}}\Big]=0$$

$$\frac{d}{dt}\Big[d\,H(x) + b\,G(y)\Big]=0 \quad \text{with} \quad \begin{aligned} H&=H(x)=\bar{x}\log x - x \\ G&=G(y)=\bar{y}\log y - y \end{aligned}$$

Define $V=V(x,y)=d\,H(x)+b\,G(y)$ for $x>0,\ y>0$

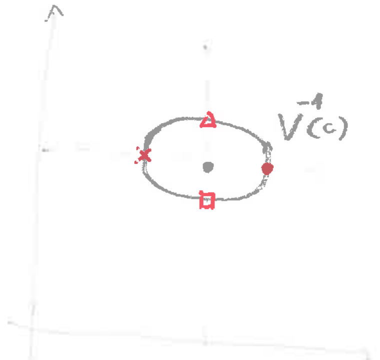
Lyapunov function

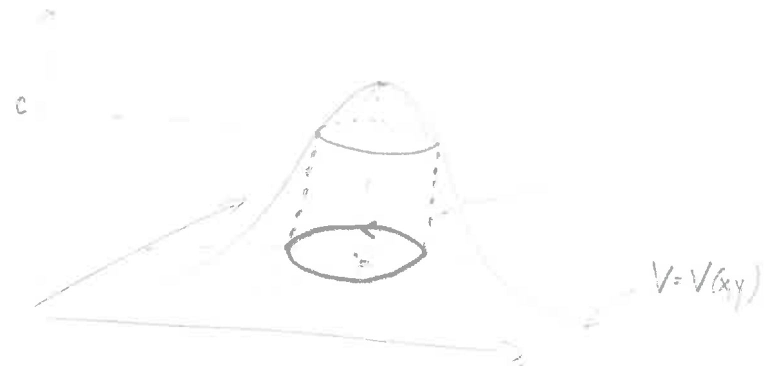
All solutions with initial condition $x_0 > 0,\ y_0 > 0$ lie on level sets of V.

$$V_{max} = V(F) \Leftarrow \begin{cases} H_{max} = H(\bar{x}) \Leftarrow \frac{d}{dx}H(\bar{x}) = 0,\ \frac{d^2}{dx^2}H(\bar{x}) < 0 \\ G_{max} = G(\bar{y}) \end{cases}$$

Also, V decrease strictly $\xrightarrow{\text{to } -\infty}$ along every line through F (check!)

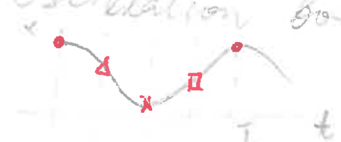
Thus, $V^{-1}(c)$ is a closed curve $\forall c$

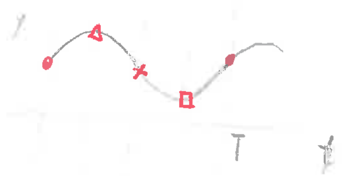




i.e. $\mathrm{int}(\mathbb{R}_+^2) \setminus F$ consists of per. orbits around F exclusively.

Interpretation: The number of predator and prey oscillate periodically with time. The amplitude and freq. of the oscillation both depend on the initial condition.





Small deviation from F $\longrightarrow$ Small amplitude
Small T / high freq.

But the time average remains the same:

$$\frac{1}{T}\int_0^T x(t)\,dt = \bar{x}, \qquad \frac{1}{T}\int_0^T y(t)\,dt = \bar{y}$$

Pf: $\frac{d}{dt}\log x = \frac{\dot{x}}{x} = a - by$

$$\Rightarrow \int_0^T \frac{d}{dt}\log x\,dt = \int_0^T (a - b\,y(t))\,dt$$

$$\Rightarrow \underbrace{\log(x(T)) - \log(x(0))}_{0} = aT - b\int_0^T y(t)\,dt$$

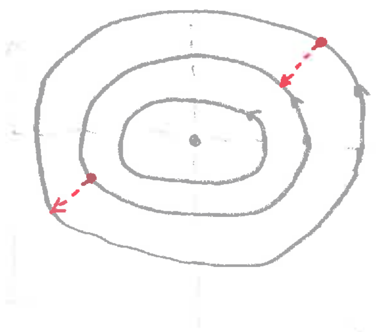
$$\Rightarrow \frac{1}{T}\int_0^T y(t)\,dt = \frac{a}{b} = \bar{y}$$

//

- Effect of fishing activity (that take place regularly)

$$\begin{cases} \dot{x} = x\,(a - k - by) \\ \dot{y} = y\,(-c - m + dx) \end{cases} \Rightarrow \quad \bar{x}_{new} = \frac{c+m}{d} \ \nearrow \qquad \bar{y}_{new} = \frac{a-k}{b} \ \searrow$$

- Effect of fishing activity (that takes place once)



fishing both

2.4 The predator-prey equation with intraspecific competition

$$\dot{x} = x(a - ex - by)$$
$$\dot{y} = y(-c - fy + dx)$$

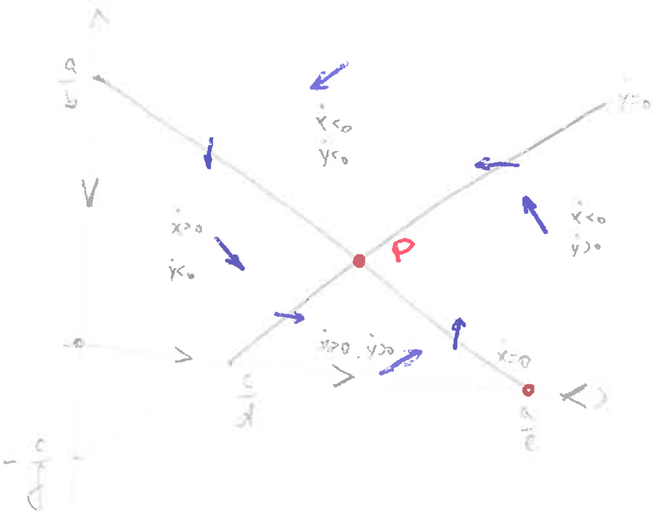
$e>0,\ f\geq 0$

in the absece of predators, the prey follows intra-competition

Two types isoclines.

(i) $\dot{x}=0 \Rightarrow x=0$ or $y = \frac{a-ex}{b}$ crosses $(0,\frac{a}{b})$, $(\frac{a}{e},0)$

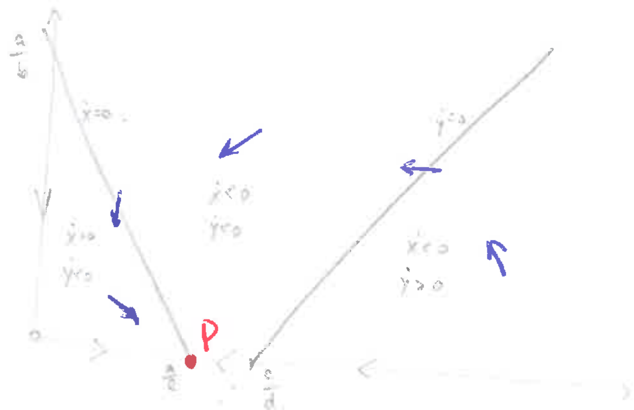
(ii) $\dot{y}=0 \Rightarrow y=0$ or $y = \frac{-c+dx}{f}$ " $(0,-\frac{c}{f})$, $(\frac{c}{d},0)$



x-axis : $\dot{x} = x(a-ex)$

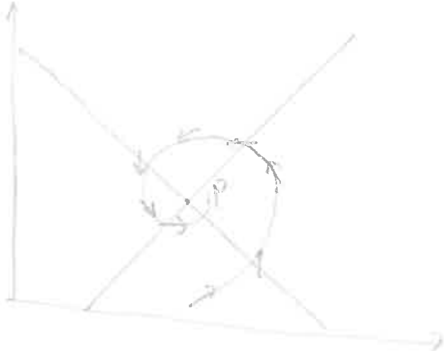
y-axis : $\dot{y} = y(-c-fy)$

per. Orbits around P ? stab. of P ?



P is a global attractor :
all orbits starting in $\mathrm{int}(\mathbb{R}_+^2)$ converge to P, as $t\to\infty$ (HW)

$\Rightarrow$ the Predator dies out.



looks like attractive p ...

we show using Lyapunov function,

first a couple of definitions.

$\dot{x} = f(x) \quad , \quad x \in \mathbb{R}^n.$

$x = x(t)$ is a solution satisfying $x(0) = x_0 \in \mathbb{R}^n$, defined for $t \geq 0$.

The asymptotics of x is contained in:

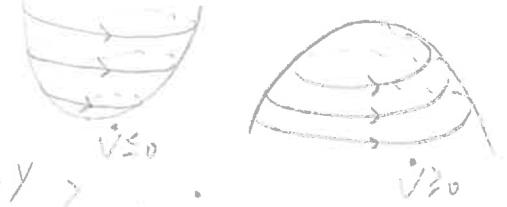
The <u>ω-limit</u> of x is the set of all accumulation points of $x(t)$, for $t \to \infty$, i.e.

$$\omega(x) = \{ y \in \mathbb{R}^n \quad x(t_k) \to y \quad \text{for some time sequence } t_k \to \infty \},$$

i.e. points whose all neighborhoods get visited by the sol. $x(t)$, even after long time. One defines <u>α-limit</u> by "$t_k \to -\infty$".

- $\omega(x)$ can be empty : $\quad \dot{x} = 1$
- $\omega(x)$ cannot be empty if the pos. semi-orb. $\{x(t) \; t \geq 0\} \subseteq K$, for some compact K
- $\omega(x)$ is closed, since $\quad \omega(x) = \bigcap_{t \geq 0} \overline{\{x(s) \; s \geq t\}}$.
- $\omega(x)$ is invariant, i.e. if $y \in \omega(x)$, then $y(t) \in \omega(x) \; \forall_{t \geq 0}$
- if $\omega(x)$ is compact, it is then also connected.
- $\omega(\text{equilibrium}) = \{\text{equilibrium}\}, \quad \omega(\text{per. sol}) = \text{per. orbit}$

Theorem (Lyapunov) Let $\dot{x} = f(x)$ be an ODE defined on $G \subseteq \mathbb{R}^n$. Let $V: G \to \mathbb{R}$ be C^1. If for some sol. $x = x(t)$, $\dot{V} := \frac{d}{dt} V(x(t))$ satisfies $\dot{V} \geq 0$ (resp. $\dot{V} \leq 0$), then $\omega(x) \cap G$ is contained in the set $\{x \in G : \dot{V}(x) = 0\}$.
The same holds for $\alpha(x) \cap G$.



Pf: $y \in \omega(x) \cap G \Rightarrow \exists\, t_k \to \infty$ s.t. $x(t_k) \to y$

$$\left.\begin{array}{l} x(t_k) \to y \\ \dot{V}(x(t)) \geq 0 \ \forall t \end{array}\right\} \Rightarrow \dot{V}(y) \geq 0$$

Assume to the contrary, $\dot{V}(y) > 0$. $\Rightarrow$ $V(y(t)) > V(y)$ $\forall t > 0$ (as initial value for $y(t)$)

$x(t_k) \to y \Rightarrow V(x(t_k)) \to V(y) \underset{V \text{ conts.}}{\overset{\dot{V} \geq 0}{\Longrightarrow}} V(x(t)) \leq V(y) \quad \forall t$

But $x(t_k + t) \to y(t) \Rightarrow V(x(t_k + t)) \to V(y(t)) > V(y)$

contradiction $\lightning$

#.

2.5. Coexistence of predator and prey

Define
$$V = V(x,y) = d\,H(x) + b\,G(y)$$
$$= d(\bar{x} \log x - x) + b(\bar{y} \log y - y) \quad \text{for } (\bar{x}, \bar{y}) = P$$

i.e. $(\bar{x}, \bar{y})$ satisfies
$$\begin{cases} a - e\bar{x} - b\bar{y} = 0 \\ -c - f\bar{y} + d\bar{x} = 0 \end{cases} \Rightarrow \begin{cases} a = e\bar{x} + b\bar{y} \\ c = -f\bar{y} + d\bar{x} \end{cases}$$

$$\frac{d}{dt} V(x(t), y(t)) = d\left(\bar{x}\frac{\dot{x}}{x} - \dot{x}\right) + b\left(\bar{y}\frac{\dot{y}}{y} - \dot{y}\right) = d\dot{x}\,\frac{\bar{x} - x}{x} + b\dot{y}\,\frac{\bar{y} - y}{y}$$
$$= d(\bar{x} - x)(a - ex - by) + b(\bar{y} - y)(-c - fy + dx)$$
$$= d(\bar{x} - x)(e\bar{x} + b\bar{y} - ex - by) + b(\bar{y} - y)(f\bar{y} - d\bar{x} - fy + dx)$$

$$= de(\bar{x}-x)^2 + d(\bar{x}-x)\cdot b(\bar{y}-y) + b(\bar{y}-y)\cdot d(x-\bar{x}) + bf(\bar{y}-y)^2$$

$$= de(\bar{x}-x)^2 + bf(\bar{y}-y)^2 \geq 0$$

$\dot{V} = 0 \overset{f>0}{\Rightarrow} (x,y) = (\bar{x},\bar{y}) = P$ for $(x,y) \in \mathrm{int}(\mathbb{R}_+^2)$

$\overset{f=0}{\Rightarrow} \{(x,y) : x=\bar{x},\ y>0\} =: \ell$

Theorem Lyapunov

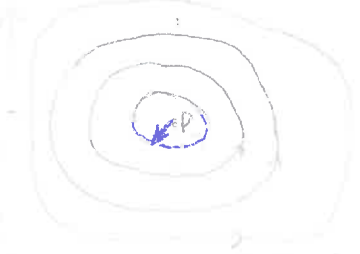
$\Longrightarrow \quad \omega(x) = \{P\} \quad \forall\, x \in \mathrm{int}(\mathbb{R}_+^2)$

(if $f=0$, $\{(x,y) : x=\bar{x}, y>0\} \supseteq \omega(x)$, but $\omega(x)$ inv. $\Rightarrow \omega(x) = \{P\}$)

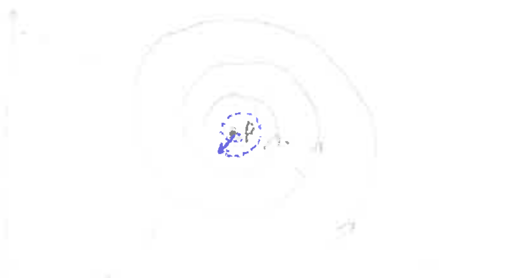
i.e. P is asym. stable, globally stable

Comparison on stability of P:

(I)



(II)



(non asym.) Stable

asym. stable

a small perturbation on P . in (I), results in per. motion,

in (II), tends back to P.

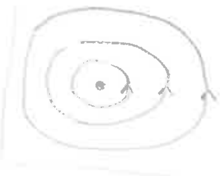
Comparison on structural stability of the system.

(I) : a small change on the vec field $\longrightarrow$ dramatic change

(II) : ... essentially the same dynamics

(Hw) perturbe the system of (I) to get an ~~ess~~ unstable "outwards spiral" equilibrium.

- predator-prey $(*_0)\begin{cases} \dot{x} = x(a-by) \\ \dot{y} = y(-c+dx) \end{cases}$



- predator-prey with intraspecific competition $(*_1)\begin{cases} \dot{x} = x(a-ex-by) \\ \dot{y} = y(-c+dx-fy) \end{cases}$

$P = (\bar{x}, \bar{y})$

$$V = d(\bar{x}\log x - x) + b(\bar{y}\log y - y) \quad \text{Lyap function}$$

$$\frac{d}{dt}V(x(t),y(t)) = de(\bar{x}-x)^2 + bf(\bar{y}-y)^2 \geq 0 \quad \forall_{(x,y)\in \operatorname{int}(\mathbb{R}^2_+)}$$

$$\xRightarrow[\text{Salle}]{\text{Lyap}} \quad \omega((x,y)) \subseteq \left\{(x,y) : \frac{d}{dt}V(x,y) = 0\right\} = \{(\bar{x},\bar{y})\} = \{P\}$$

pos. orb. enclosed in a cpt set $\Rightarrow \omega((x,y)) \neq \phi$

$$\Longrightarrow \quad \omega((x,y)) = \{P\} \quad \forall_{(x,y)\in \operatorname{int}(\mathbb{R}^2_+)}$$

$\Rightarrow$ P is glob. asym. stable

Rmk: (structural stability) A C^r vec. field f is s. stable if

$\exists_{\varepsilon>0}\ \forall$ C^1-ε-perturbation $\tilde{f}$ of f : $\tilde{f} \sim f$ are topo. equivalent,

$\left\langle \begin{array}{l} |f-\tilde{f}|<\varepsilon, \left|\frac{\partial f}{\partial x_i} - \frac{\partial}{\partial x_i}\tilde{f}\right| < \varepsilon \text{ within } K \\ f = \tilde{f} \text{ outside } K \end{array}\right.$

$\psi_t{}_{A\to A}$ $\varphi_t{}_{B\to B}$ flows gene. by $\tilde{f}$, f

$\left[\begin{array}{l} \frac{d}{dt}\varphi(x,t)\Big|_{t=t} = f(\varphi \ldots) \\ \frac{d}{dt}(\psi(x,t))\Big|_{t=\tau} = \tilde{f}(\psi(x,\tau)) \end{array}\right.$

$\exists$ homeom. $h: A\to B$,

$\exists$ $\tau: A\times\mathbb{R}\to\mathbb{R}$ increasing on $\mathbb{R}$ s.t.

$$h(\varphi_{\tau(x,t)}(x)) = \psi_t(h(x))$$

i.e. h maps orbits to orbits.

$(*_0)$ is not s. stable, since $(*_1)$ is a perturbation and $(*_1) \not\sim (*_0)$.

More on s. stability: linearization and Thm Hartman-Grobman

- linear flows

(L) $\dot{x} = Ax$, $x \in \mathbb{R}^n$, $A \in M_{n \times n}$ generates linear flows on $\mathbb{R}^n$ by

$x(t) = e^{At} \cdot x(0)$, where $e^{At} = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \dots$

Solutions are lin. comb. of

(i) $e^{\lambda t}$, if $\lambda \in \sigma(A) \cap \mathbb{R}$

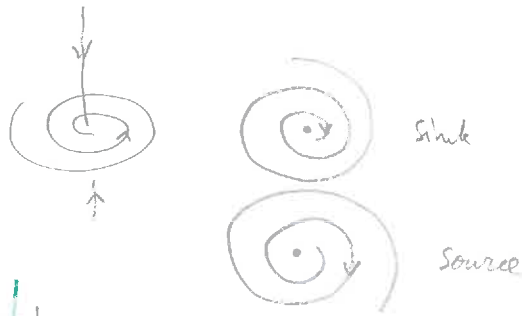
(ii) $e^{at}\cos bt$ and $e^{at}\sin bt$ if $a \pm ib \in \sigma(A)$

(iii) $t^j e^{\lambda t}$ or $t^j e^{at}\cos bt$ and $t^j e^{at}\sin bt$ with $0 \le j < m$, if λ or $a \pm ib$ occur with multip. m.

$x=0$ of (L) is called

(a) a sink, if $Re(\lambda) < 0 \ \forall_{\lambda \in \sigma(A)}$

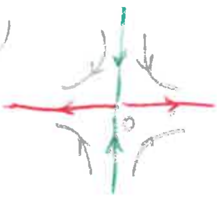
(b) a source, if $Re(\lambda) > 0 \ \forall_{\lambda \in \sigma(A)}$



(c) a saddle, if $Re(\lambda) > 0$ for some $\lambda \in \sigma(A)$
$Re(\lambda) < 0$ for the rest.

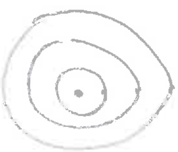
stab mnf $= \{x : \omega(x) = \{0\}\}$

unstab mnf $= \{x : \alpha(x) = \{0\}\}$



The rest point 0 is called hyperbolic, if (a), (b) or (c), i.e. $Re(\lambda) \neq 0 \ \forall_{\lambda \in \sigma(A)}$

$\lambda = 0$



$\lambda = \pm ib$

$\dot{x} = 0$

vs.

$\dot{x} = \pm\varepsilon$

$\dot{x} = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} x$

vs

$\dot{x} = \begin{pmatrix} \pm\varepsilon & -b \\ b & \pm\varepsilon \end{pmatrix} x$

- linearization :

(N) $\dot{x} = f(x)$ $\qquad$ x near $z \in \mathbb{R}^n$ $\qquad$ Taylor $f(x) = f(z) + Df(z)(x-z) + h.o.t.$

(a) z is not an equilibrium

$\boxed{\text{flow box thm}}$.

$\dot{z}$ up to coords. transf.

(b) z is an equilibrium

$$f(x) = \underset{\substack{\parallel \\ 0}}{f(z)} + \underbrace{\boxed{Df(z)(x-z)}}_{\text{dominates}} + h.o.t.$$

$\dot{x} = f(x)$, x near z $\qquad$ nonlin.

???

$\dot{x} = Df(z)(x-z)$ for equiv. $\dot{y} = Df(z)\,y$, $(y = x-z)$ $\qquad$ lin.

- Thm (Hartman-Grobman) Let z be a hyperb. equilibrium of (N).

Then $\exists$ nbhd U of z, a nbhd V of 0 and a homeom.

$$h : U \to V, \quad h(z) = 0$$

s.t. $y = h(x)$ implies $y(t) = h(x(t))$ $\forall t \in \mathbb{R}$, $x = x(t)$ solves $\dot{x} = f(x)$,
$y = y(t)$ solves $\dot{y} = Df(z)\,y$.

i.e. $\exists$ homeom. h s.t. $\underbrace{\psi_t = h \circ \varphi_t \circ h^{-1}}_{\text{"topo conj"}}$ locally around z

equivalent flows.

Hw every flow is locally around a hyperb. equilibrium s. stable

• Competition equation

$$(*) \quad \begin{cases} \dot{x} = x(a - bx - cy) \\ \dot{y} = y(d - ex - fy) \end{cases}$$

d/f $\vee$ • $\wedge$?

$> \bullet <$ a/b

isoclines: $a - bx - cy = 0$, $d - ex - fy = 0$, if they

(a) coincide, i.e. $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$ then

HW every solu satisfies $\frac{d}{dt}\left(x(t)\,y(t)^{-k}\right) = 0$ $\forall t \in \mathbb{R}$, $k = \frac{a}{d}$, i.e. xy^{-k} is const of motion

(b) not coincide:

(b1) not intersect in $\mathrm{int}(\mathbb{R}^2_+)$: one species tends to extinct

(b2) intersect in $\mathrm{int}(\mathbb{R}^2_+)$

stable

(I)

bistable

(II)

Using linearization around $P = (\bar{x}, \bar{y}) = \left(\frac{af - cd}{bf - ce}, \frac{bd - ae}{bf - ce}\right)$

$$A = \begin{pmatrix} -b\bar{x} & -c\bar{x} \\ -e\bar{y} & -f\bar{y} \end{pmatrix} \qquad \det A = (bf - ce)\bar{x}\bar{y}$$

- $bf > ce$ $\left(\Rightarrow af > cd,\ bd > ae \Rightarrow \frac{b}{e} > \frac{a}{d} > \frac{c}{f} \Rightarrow \frac{a}{b} < \frac{d}{e} \Rightarrow (\mathrm{I})\right)$

  $\det A > 0 \xRightarrow{tr A < 0} \lambda_1, \lambda_2 < 0 \Rightarrow P$ sink $\Rightarrow$ stable

- $bf < ce$ $\left(\Rightarrow \frac{c}{f} > \frac{a}{d} > \frac{b}{e} \Rightarrow \frac{a}{b} > \frac{d}{e} \Rightarrow (\mathrm{II})\right)$

  $\det A < 0 \xRightarrow{tr A < 0} \lambda_1 > 0, \lambda_2 < 0 \Rightarrow P$ saddle $\Rightarrow$ unstable
  (in fact, bistable)

Hw $Q(x,y) = be(x-\bar{x})^2 + 2ce(x-\bar{x})(y-\bar{y}) + cf(y-\bar{y})^2$ is a Lyapunov function for $(*)$.

- Cooperative systems

$$\begin{cases} \dot{x} = x(a - bx + cy) \\ \dot{y} = y(d + ex - fy) \end{cases}$$

or more generally

Def $\dot{x} = f(x)$ on $G \subseteq \mathbb{R}^n$ is called cooperative, if

$\frac{\partial f_i}{\partial x_j}(x) \geq 0 \quad \forall x \in G, \ i \neq j$.

⌈The growth of every species is enhanced by an increase of any other species.⌋

Thm: (2D-cooperative system) The orbits of a 2D-cooperative system converge either to a rest point or to infinity

pf:

- $\dot{x}(t_0) \in C_1 \Rightarrow \dot{x}(t) \in C_1 \quad \forall_{t \geq t_0}$ $\qquad \dot{x}_1 > 0 \Leftrightarrow x_1 \nearrow \Leftrightarrow x_2 \nearrow \Leftrightarrow \dot{x}_2 > 0$
- $\dot{x}(t_0) \in C_3 \Rightarrow \dot{x}(t) \in C_3 \quad \forall_{t \geq t_0}$
- $\dot{x}(t_0) \in C_2 \Rightarrow \dot{x}(t) \in C_2 \quad \forall_{t \geq t_0}$

  or $\dot{x}(t) \in C_1$ or C_3 after finite time
- $\dot{x}(t_0) \in C_4$ similar to C_2

$\Rightarrow$ $x_1 = x_1(t)$, $x_2 = x_2(t)$ are ultimately monotonic

$\Rightarrow$ converge to a limit if not infinity

//.

The same holds for competitive system, $\frac{\partial f_i}{\partial x_j} \leq 0 \quad \forall_{i \neq j}$

4. Ecological equations for two species

On periodic orbits

Thm (Poincaré-Bendixson) A nonempty compact ω- or α-limit set of a planar flow, which contains no equilibria, is a closed orbit. (on open set $G \subseteq \mathbb{R}^2$)

Consequently,

- $K \subseteq G$ nonempty, compact, forward invariant $\Rightarrow$ contains an equil. or a per. orbit.
-  $\Rightarrow$ Γ contains an equilibrium

Thm (Bendixson-Dulac) If on a simply conn. region $G \subseteq \mathbb{R}^2$, the expression $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is non-identically zero, and does not change sign, in G, then G contains no per. orbits.

pf:

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \Rightarrow \frac{dy}{dx} = \frac{g}{f} \quad \text{or} \quad \int_{\partial} \left(f(x,y)\,dy - g(x,y)\,dx \right) = 0 \quad \forall \text{ closed } \gamma$$

$$\overset{\text{Green's}}{\Longrightarrow} \iint_S \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dx\,dy = 0 \quad \text{but} \quad \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} > 0 \text{ entirely on } G \text{ or } < 0$$

$$\Rightarrow S \not\subseteq G$$

$\boxtimes$

Two-dim'l Lotka-Volterra

$$(*)\begin{cases}\dot{x} = x(a+bx+cy) = P \\ \dot{y} = y(d+ex+fy) = Q\end{cases} \qquad a,b,c,d,e,f \in \mathbb{R}$$

Thm. The above has no isolated per. orbits.

Pf: Assume γ is a per. orbit of $(*)$ in $\mathbb{R}^2_+$

$\Rightarrow$ γ contains an equilibrium

$$\begin{cases} a+bx+cy = 0 \\ d+ex+fy = 0 \end{cases} \quad \text{intersect uniquely at } F = \left(\frac{-af+cd}{bf-ce}, \frac{-bd+ae}{bf-ce}\right)$$

$\Rightarrow$ $\Delta = bf - ce \neq 0$

Using the Dulac function $B(x,y) = x^{\alpha-1}y^{\beta-1}$ for some α, β

$$\mathrm{div}(BP, BQ) = \frac{\partial}{\partial x}(BP) + \frac{\partial}{\partial y}(BQ) = \frac{\partial}{\partial x}\left(x^{\alpha}y^{\beta-1}(a+bx+cy)\right) + \frac{\partial}{\partial y}\left(x^{\alpha-1}y^{\beta}(d+ex+fy)\right)$$

$$= \alpha x^{\alpha-1}y^{\beta-1}(a+bx+cy) + bx^{\alpha}y^{\beta-1} + \beta x^{\alpha-1}y^{\beta-1}(d+ex+fy) + f x^{\alpha-1}y^{\beta}$$

$$= B\left(\alpha(a+bx+cy) + bx + \beta(d+ex+fy) + fy\right)$$

$$= B\Big(a\alpha + d\beta + \underbrace{(\alpha b + b + \beta e)}_{=0}x + \underbrace{(\alpha c + \beta f + f)}_{=0}y\Big)$$

choose α, β s.t. ... i.e. $\alpha = \frac{-bf+ef}{bf-ce}$ $\beta = \frac{-bf+bc}{bf-ce}$

$$= B(\underbrace{a\alpha + d\beta}_{=:\delta})$$

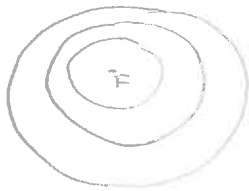
$\xRightarrow[\exists \text{ per. Orb. } \gamma]{\text{Bendixson-Dulac}}$ $\delta = a\alpha + d\beta = 0 \quad \wedge \quad \frac{\partial}{\partial x}(BP) = -\frac{\partial}{\partial y}(BQ)$

integrability

$\Longrightarrow$ $\exists\, V = V(x,y)$ s.t. $\frac{\partial V}{\partial x} = BQ$, $\frac{\partial V}{\partial y} = -BP$

Hw $V = x^p y^q (A + Bx + Cy)$ for some $p, q, A, B, C \in \mathbb{R}$.

$\Rightarrow \dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = PQ(B-B) \equiv 0$



continuum

$\Rightarrow$ γ is non-isolated

#

Corollary L-V in 2D does not admit $\leq$ stable limit cycles

Def. A per. orb γ is called an attractor if $\omega(x) = \gamma$ $\forall$ init. cond. x in some nbhd of γ

A per. orb. γ is called a limit cycle, if $\omega(x) = \gamma$ for at least $x \notin \gamma$.

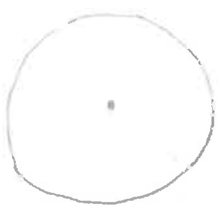
Example
$$\begin{cases} \dot{x} = x - y - x(x^2+y^2) \\ \dot{y} = x + y - y(x^2+y^2) \end{cases}$$
admits a per. attractor $\underbrace{\{(x,y) \; x^2+y^2=1\}}_{S}$

$V = V(x,y) = (1-x^2-y^2)^2$ $\quad \frac{\partial V}{\partial x} = -4x(1-x^2-y^2)$ $\quad \frac{\partial V}{\partial y} = -4y(1-x^2-y^2)$

$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = -4x(1-x^2-y^2)\left((1-x^2-y^2)x - y\right) - 4y(1-x^2-y^2)\left((1-x^2-y^2)y + x\right)$

$= -4x^2(1-x^2-y^2)^2 + 4xy(1-x^2-y^2) - 4y^2(1-x^2-y^2)^2 - 4xy(1-x^2-y^2)$

$= -4(x^2+y^2)(1-x^2-y^2)^2 \leq 0$



0 is unstable. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$tr = 2 \quad \Delta = 2 > 0. \quad \lambda_{1,2} > 0.$ //.

5. Stable limit cycle ⟿ nonlinearity

Predator-Prey model of Gauss (nonlin. model)

$$\begin{cases} \dot{x} = x g(x) - y p(x) \\ \dot{y} = y(-d + q(x)) \end{cases}$$

where

(a) $g(x) > 0$ for $x < K$

$g(x) < 0$ for $x > K$

(b) $p(x) = $ # prey killed per predator

$p(0) = 0$, $p(x) > 0$ for $x > 0$

(c) $q(x) = $ # predator nurtured per prey

$q(0) = 0$, $q(x) > 0$ for $x > 0$, $q'(x) > 0$ for $x > 0$

(I)

x-isocline $y = \frac{x g(x)}{p(x)}$

y-isocline $x = \bar{x}$

predator extincts!

(II)

Consider (I) now

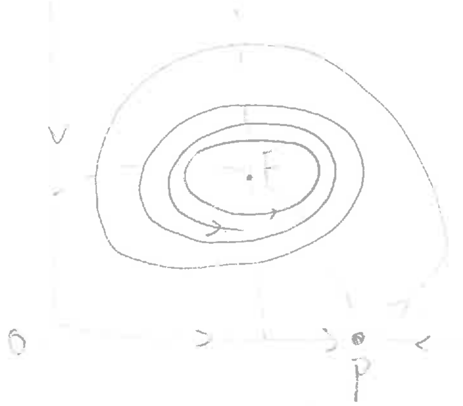
$O=(0,0)$, $P=(K,0)$, F

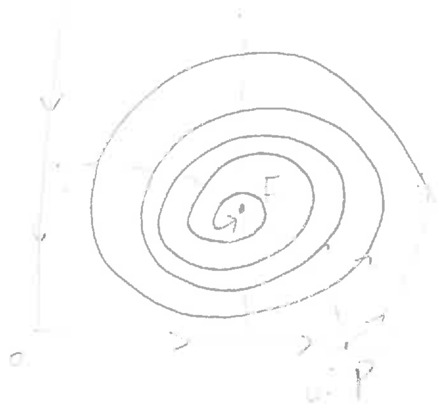
$$J_{(x,y)} = \begin{pmatrix} g(x)+xg'(x)-yp'(x) & -p(x) \\ yf'(x) & -d+f(x) \end{pmatrix}$$

- $J_{(0,0)} = \begin{pmatrix} g(0)^{>0} & 0 \\ 0 & -d \end{pmatrix}$ in fact, $(0,0)$ is a saddle

- $J_{(K,0)} = \begin{pmatrix} \underbrace{Kg'(K)}_{<0} & -p(K) \\ 0 & \underbrace{-d+f(K)}_{>0} \end{pmatrix}$ saddle

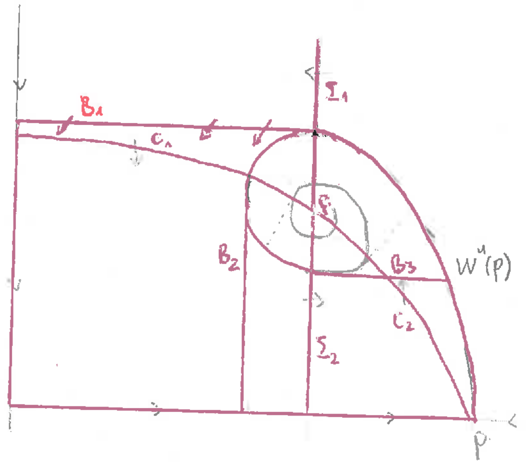
$f(\bar{x})=d$, $\bar{x}<K$





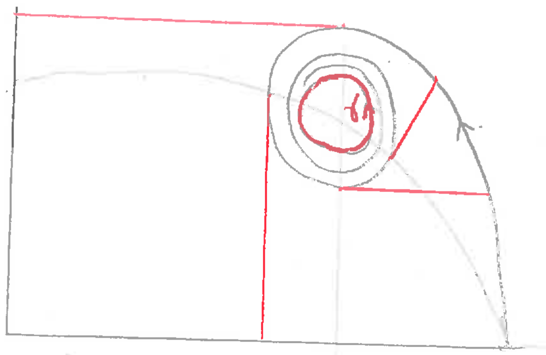
if $\omega(x)$ contains no equilibria, then it is a per. orbit

if $\omega(x)$ contains equilibria (it has to be F), then F is stable



- $W^u(p) \cap \Sigma_1$ transverse $\dot{x} < 0$ $\dot{y} > 0$ $y \to$
- Barrier B_1, $W^u(p) \cap C_1$ transverse $\dot{x} < 0$ $x \to 0$ $\dot{y} < 0$

F is globally stable.



Similarly, γ is "onesided" stablle.

Stability of per. Orbits and Poincaré Map

- Poincaré map

- $S \cap \gamma = \{x_0\}$
- $S \cap$ flow around γ transversely,

i.e.

$f(x) \cdot \underbrace{\nu_x(S)}_{\text{normal unit vector}} \neq 0 \quad \forall x \in S$

$P \quad S \longrightarrow S$

$x \longmapsto \varphi_{\tau(x)}(x)$, $\tau(x)$ = first return time for x

S is called Poincaré section;

P " " Poincaré map.

<u>Existence</u> of (P, S) is guaranteed by cont. of f, φ, and conts dependence of initial value.

<u>Significance</u> : stability of per. orbits using (P, S)

<u>Def.</u> : A cpt inv. set K is called <u>stable</u>, if

$$\forall_{\text{nbhd } U \text{ of } K} \ \exists_{\text{nbhd } V \text{ of } K} \quad \text{s.t.} \quad O^+(x) \subseteq V \quad \forall_{x \in U}$$

It is called <u>attractive</u>, if

$$\exists_{\text{nbhd } U \text{ of } K} \quad \text{s.t.} \quad \omega(x) \subset K \quad \forall_{x \in U}$$

A closed orbit is stable or attractive, if it is so as a cpt inv. set; it is <u>asymptotically stable</u>, if it is both stable and attractive.

<u>Theorem (Poincaré)</u> Let γ be a per. orb. of $\dot{x} = f(x)$, for $f : G \subseteq \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz conts. Let $x_0 \in \gamma$ and $P : S \to S$ be a Poincaré map on S around x_0. Then,

(i) γ is stable $\Longleftrightarrow$ x_0 is stable as a f.p. of P,

i.e. $\forall_{\varepsilon>0} \exists_{\delta>0}$ s.t. $P^n(x)$ is defined for $|x-x_0|<\delta$

and $|P^n(x) - x_0| < \varepsilon$ $\forall n$.

(ii) γ is asym. stable $\Leftrightarrow$ x_0 is asym. stable as a f.p of P,

i.e. $\exists_{\delta>0}$ s.t. $P^n(x)$ is def. for $|x-x_0|<\delta$

and $P^n(x) \to x_0$ as $n\to\infty$.

(iii) γ is unstable $\Longleftrightarrow$ x_0 is an unstable f.p. of P.

i.e. $\exists_{x\in S}$ s.t. $P^{-n}(x)$ is def. and $P^{-n}(x) \to x_0$ as $n\to\infty$

Pf: "$\Rightarrow$" (HW)

"$\Leftarrow$" Conts. dependence of initial value and opt. of σ #.

---

5. Stability for flows in $\mathbb{R}^2$ or M^2.

Theorem (Andronov-Pontryagin) A vec field in $\mathbb{R}^2$ is s. stable iff:

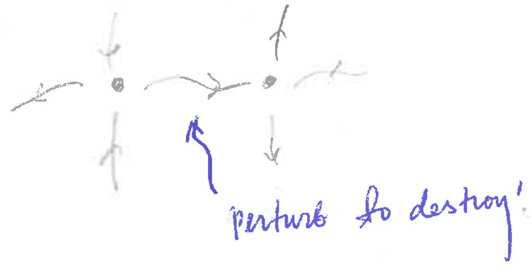
(i) all its f.p.s (equilibria) are hyperbolic

(ii) all its per. orbits are hyperbolic

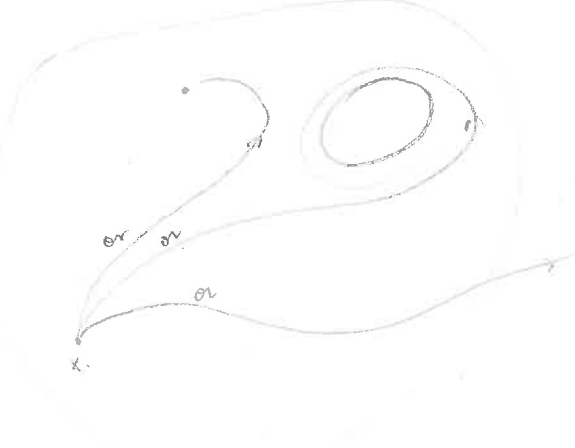
(iii) no connecting trajectories from saddle to saddle.

pf: "⇒" (i) (ii) ✓ vec. fd with non-hyperbolic f.p. or per Orb. can NOT be s.stable

(iii) Saddle connection



"⇐" Poincaré Bendixson: $\omega(x)$ either contains a f.p. or is a closed orbit.



Theorem (Peixoto) A C^r-vec fd on a cpt 2d M^2 is s. Stable. iff (i)–(iii) of A-P hold and

(iv) the nonwandering set consists of f.p.s and per. orb. alone.

Def. A point x is called nonwanding if

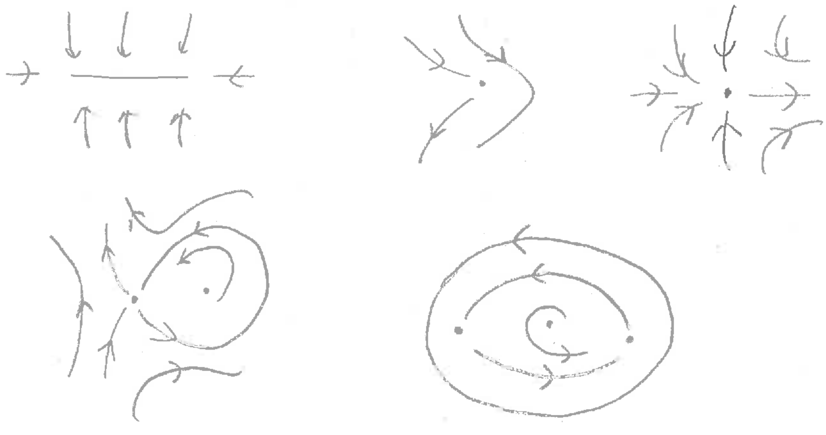
$$\forall_{\substack{\text{nbhd } U \\ \text{of } x}} \ \forall_{T>0} \ \exists_{t>T} \ \text{s.t.} \ \varphi_t(U) \cap U \neq \emptyset .$$

The nonwandering set is the set of all nonwandering points

Moreover, if M^2 is orientable, then

$$\{\text{ s.stable } C^r \text{ v.f. on } M^2 \} \underset{\substack{\text{open}\\ \text{dense}}}{\subseteq} \{ C^r\text{-v.f. on } M^2 \}$$

<u>Examples</u>:



<u>Def.</u> A <u>Morse-Smale</u> System is a system for which

(i) it has fin. many f.p. and/or per. orbits, all hyperbolic

(ii) all stable and unstable mfds intersect transversely

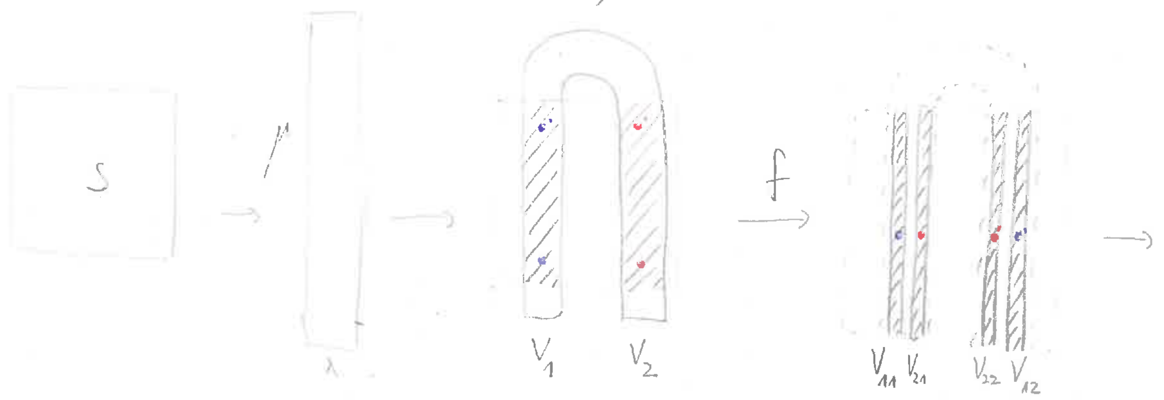
(iii) the non-wandering set consists of f.p. and per. orb alone.

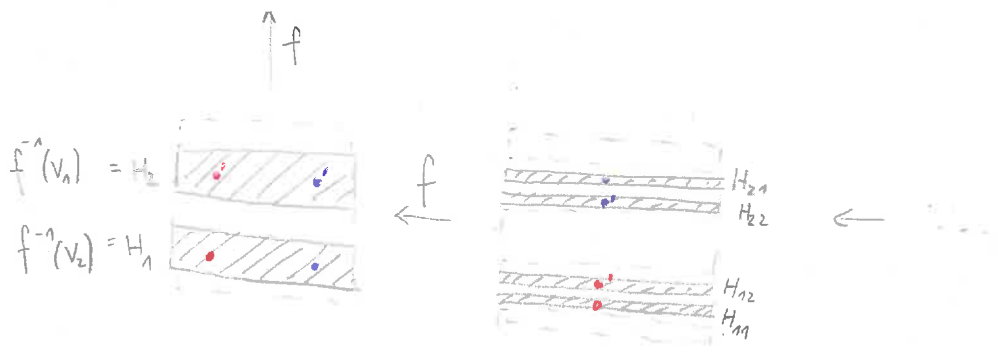
<u>Remark</u>: (i) $\text{Morse-Smale} \underset{\not\Longleftarrow \text{ for } n>2}{\Longrightarrow} \text{s. stable}$

(ii)
$$\begin{array}{l} \text{M-S} \subseteq \\ \text{s. stable} \subseteq \end{array} \text{ not dense in } \{\text{all } C^r\text{-vec fld.}\}$$

if $n>2$

Example (Horse-Shoe Map)





$$\begin{cases} V_1 \cup V_2 = S \cap f(S) \\ H_1 \cup H_2 = f^{-1}(S \cap f(S)) = f^{-1}(S) \cap S \end{cases}$$

$$V_i = f(H_i)$$

$$\begin{cases} V_{11} \cup V_{21} \cup V_{22} \cup V_{12} = S \cap f(S) \cap f^2(S) \\ H_{21} \cup H_{22} \cup H_{12} \cup H_{11} = f^{-2}(S \cap f(S) \cap f^2(S)) = f^{-2}(S) \cap f^{-1}(S) \cap S \end{cases}$$

$$V_{ij} = f^2(H_{ij})$$

$$\Lambda = \bigcap_{i=-\infty}^{\infty} f^i(S)$$

$V_1 \cap H_2$	12	22	$V_2 \cap H_2$
$V_1 \cap H_1$	11	21	$V_2 \cap H_1$

$x \in V_i \cap H_j \Leftrightarrow x \in f(H_i) \cap H_j$

$\Leftrightarrow f^{-1}(x) \in H_i \wedge x \in H_j$

$V_{11} \cap H_{21}$	1121	2121	2221	1221	$V_{12} \cap H_{21}$
	1122	2122	2222	1222	
$\vdots$					
	1112	2112	2212	1212	
$V_{11} \cap H_{11}$	1111	2111	2211	1211	$V_{12} \cap H_{11}$

$x \in V_{ij} \cap H_{kl} \Leftrightarrow x \in f^2(H_{ij}) \cap H_{kl}$

$\Leftrightarrow x \in H_k \wedge f(x) \in H_l \wedge$

$f^{-2}(x) \in H_i \wedge f^{-1}(x) \in H_j$

$$\Sigma = \{\text{bi-infinite sequences in two symbols}\} = \{ a = \{a_i\}_{i=-\infty}^{\infty} : a_i \in \{1,2\} \}$$

$$d(a,b) = \sum_{i=-\infty}^{\infty} \delta_i \, 2^{-|i|}, \quad \delta_i = \begin{cases} 0 & a_i = b_i \\ 1 & a_i \neq b_i \end{cases} \quad a, b \in \Sigma.$$

$$
\begin{array}{ccc}
\Lambda & \xrightarrow{\ \phi\ } & \Sigma \\
x & \longmapsto & \phi(x) = \ldots 2112\dot{1}21122 \ldots \quad \text{as above} \\
\Big\downarrow f & & \text{if} \quad \ldots \ f^{-1}(x) \in H_1 \wedge x \in H_2 \wedge f(x) \in H_1 \wedge \ldots \\
f(x) & \longmapsto & \phi(f(x)) = \ldots 11212\dot{1}1122 \ldots \\
& & \text{since} \quad \ldots \ f(x) \in H_1 \wedge f^2(x) \in H_1 \wedge \ldots
\end{array}
$$

i.e. $\Lambda \xrightarrow{f} \Lambda$ corresponds to $\Sigma \xrightarrow{s} \Sigma$ the 1-shift to the left.

Indeed, ϕ defines a homeom. between Λ and Σ. Thus, to understand (Λ, f), it is enough to look at (Σ, s).

- There are infinitely many per. orbits
- Periodic orbits of arb. long per. exist.
- The set of per. orbits is a dense set in Λ
- $\exists$ a dense orbit in Λ ([illegible])

$\Rightarrow$ Λ is not M-S, however s. stable.

#

Bifurcation Theory (local around equilibria)

(*) $\quad \dot{x} = f(\mu, x), \quad x \in \mathbb{R}^n, \ \mu \in \mathbb{R}^k, \ f$ smooth

We say that (*) undergoes a bifurcation when $\mu = \mu_0$ if

(a) The flow of (*) for $\mu = \mu_0$ is not structurally stable; or

* (b) In every nbhd of μ_0 in $\mathbb{R}^k$ there is a μ s.t. $\varphi_\mu \not\sim \varphi_{\mu_0}$ non topo equivalent

[Remark: (a) considers all v.f. in close nbhd of $f(\mu_0, x)$; (b) only those of form $f(\mu, x)$.

i.e. bf in sense of (a) $\underset{\not\Rightarrow}{\Leftarrow}$ bf in sense of (b)

Ex. $\quad \dot{x} = -(\mu^2 x + x^3) \quad \mu = 0$ is bf. value in sense of (a) but not in sense of (b)]

*.

Bifurcation of Equilibria (Steady-state bf.)

Equilibria set of (*) $= \{(\mu, x) : f(\mu, x) = 0\} = E$

For $(\mu_0, x_0) \in E$ with $D_x f(\mu_0, x_0)$ invertible, there is a nbhd of (μ_0, x_0) s.t.

(I.F.T.)

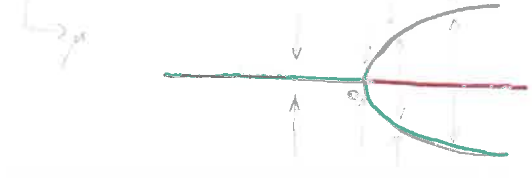
$E \cap B(\mu_0, x_0)$ can be smoothly parametrized by μ,

i.e. unique branch of equilibria as μ moves through μ_0

Otherwise, there can be multiple branches of equilibria "bifurcating" from $\mu = \mu_0$

Ex. $\quad \dot{x} = \mu x - x^3, \quad \mu, x \in \mathbb{R}$

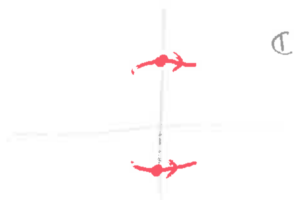
$E = \{(\mu, x) \in \mathbb{R}^2 : \mu x - x^3 = 0\}$ $\qquad \mu x - x^3 = 0 \Rightarrow x(\mu - x^2) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{\mu}$ if $\mu > 0$



$D_x f(0,0) = \mu$

Bifurcation of Periodic Orbits (Andronov-Hopf bif)

Another way an equilibrium can change its stability is



$D_x f(\mu_0, x_0)$ is invertible but $\pm i\beta \in \sigma(D_x f(\mu_0, x_0))$ for some $\beta > 0$.

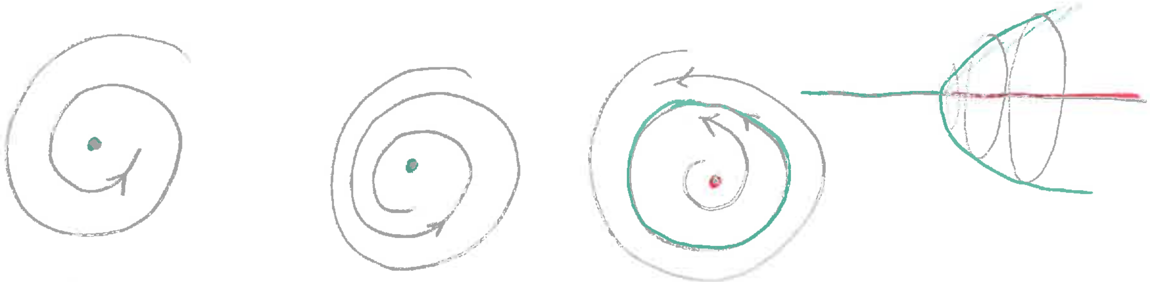
By I.F.T., equilibria around (μ_0, x_0) remain parametrized uniquely by μ, however other type of solutions may arise.

Ex.
$$\begin{cases} \dot{x} = \mu x - y - x(x^2+y^2) \\ \dot{y} = x + \mu y - y(x^2+y^2) \end{cases}$$
$(0,0)$ is an equilibrium $\forall \mu$.

$D_{(x,y)} f(\mu, (0,0)) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$ has eigenvalues $\mu \pm i$

Polar coordinates: $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$

$$\begin{cases} \dot{r} = (\mu - r^2) r \\ \dot{\theta} = 1 \end{cases}$$
$r = 0$ or $r = \sqrt{\mu}$ for $\mu > 0$



$\mu < 0$ asy stable

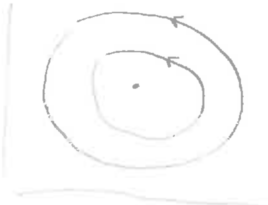
$\mu = 0$ (weakly) asym. stable

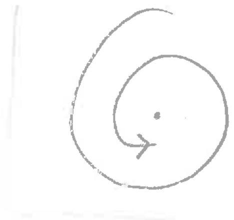
$\mu > 0$

Also recall

$$(*_0)\begin{cases}\dot{x} = x(a-by)\\ \dot{y} = y(-c+dx)\end{cases} \qquad \text{vs.} \qquad (*)\begin{cases}\dot{x} = x(a-ex-by)\\ \dot{y} = y(-c+dx-fy)\end{cases}$$

L-V predator-prey $\qquad\qquad$ L-V p-p with intraspecific competition





One can view $(*_0)$ as a special case of $(*)$ for $e=f=0$

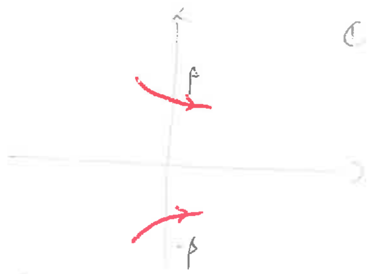
Consider $(*)$ and its equilibrium $(\bar{x},\bar{y}) \in \operatorname{int}(\mathbb{R}^2_+)$, i.e.

$$\begin{cases} a - e\bar{x} - b\bar{y} = 0 \\ -c + d\bar{x} - f\bar{y} = 0\end{cases} \qquad D_{(x,y)}f(\bar{x},\bar{y}) = \begin{pmatrix} a-2e\bar{x}-b\bar{y} & -b\bar{x} \\ d\bar{y} & -c+d\bar{x}-2f\bar{y}\end{pmatrix} = \underbrace{\begin{pmatrix} -e\bar{x} & -b\bar{x} \\ d\bar{y} & -f\bar{y}\end{pmatrix}}_{= L}$$

$$\Rightarrow \begin{cases} \operatorname{tr} L = -e\bar{x} - f\bar{y} = \lambda_1 + \lambda_2 \\ \det L = ef\,\bar{x}\bar{y} + bd\,\bar{x}\bar{y} = \lambda_1\lambda_2 \end{cases}$$

For convenience, let $e=f$ and allow $e \in \mathbb{R}$. Then

- $e=0$ $\qquad \operatorname{tr} L = 0$, $\det L > 0$ $\qquad \Rightarrow \lambda_{1,2} = \pm i\beta$ for $\beta^2 = bd\bar{x}\bar{y}$
- $e>0$ $\qquad \operatorname{tr} L < 0$, $\det L > 0$
- $e<0$ (close to 0) $\qquad \operatorname{tr} L > 0$, $\det L > 0$



$e<0$ unstable $\qquad e=0$ stable $\qquad e>0$ asym. stable

Bifurcation from an equilibrium $x_0 \Rightarrow \sigma(D_x f(\mu_0, x_0)) \cap i\mathbb{R} \neq \phi$
for $\mu = \mu_0$

(change of stability)

Moreover,

- bifurcation of steady states $\overset{\not\Leftarrow}{\Rightarrow} \quad 0 \in \sigma(D_x f(\mu_0, x_0))$ $\quad \dot{x} = -\mu^2 x - x^3$
- bifurcation of per. states $\overset{\not\Leftarrow}{\Rightarrow} \quad \pm i\beta \in \sigma(D_x f(\mu_0, x_0))$ (Hw) $\begin{cases} \dot{r} = -\mu^2 r - r^3 \\ \dot{\theta} = 1 \end{cases}$

We call 0 (resp. $\pm i\beta$) critical eigenvalues of $D_x f(\mu_0, x_0)$ for $\mu = \mu_0$

Codimension of Bifurcation

Depending on the type of critical eigenvalues, we define codimensio

Lemma Let $M_{2\times 2}$ be the set of all 2×2 real matrices. ($M_{2\times 2} \cong \mathbb{R}^4$)
Then, $S = \{A \in M_{2\times 2} \quad A$ has 0 as a simple eigenvalue$\}$ is a submanifold
of codimension 1 in $M_{2\times 2}$

Pf: $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det(\lambda I - A) = \lambda^2 - (a+d)\lambda + ad - bc = 0 \quad \Rightarrow \begin{cases} \lambda_1 + \lambda_2 = a+d \\ \lambda_1 \lambda_2 = ad - bc \end{cases}$

$\lambda = 0$ is an eigenvalue of $A \Rightarrow 0 = \lambda_1 \lambda_2 = ad - bc$

$\lambda = 0$ is simple $\Rightarrow \lambda_1 + \lambda_2 \neq 0 \Rightarrow a + d \neq 0$

Define $\quad F: \quad M_{2\times 2} \longrightarrow \mathbb{R} \qquad\qquad S \subseteq F^{-1}(0)$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \det A = ad - bc$$

$DF(A) = (d \;\; -c \;\; -b \;\; a) \qquad$ For $A \in S \Rightarrow a + d \neq 0 \Rightarrow \operatorname{rank} DF(A) = 1$

$\Rightarrow \forall A \in S$ is a regular point of F

$\Rightarrow S$ is a submanifold of dim 3 of $M_{2\times 2} \Rightarrow \operatorname{codim} S = 1$

Def Suppose (*) undergoes a bifurcation of equilibria at $\mu = \mu_0$ and $D_x f(\mu_0, x_0)$ has critical eigenvalues of type (T) (such as "it has 0 as a simple eigenvalue")

Then, we say the bifurcation is of codimension m, if

$$\{ A \in M_{n\times n} \quad A \text{ has eigenvalues of type } (T) \} \subseteq M_{n\times n}$$

is a submfd of codimension m

Examples.

- Codimension One.

(i) simple zero eigenvalue $\quad D_x f_\mu = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$

(ii) simple purely imag. pair e.v. $D_x f_\mu = \begin{pmatrix} \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} & 0 \\ 0 & A \end{pmatrix}$

- Codimension Two.

(iii) double zero, nondiagonalizable $\quad D_x f_\mu = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & A \end{pmatrix}$

(iv) simple zero, simple pair pure imag $\quad \begin{pmatrix} \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & 0 \\ 0 & A \end{pmatrix}$

(v) two distinct pairs pure imag. $\quad \begin{pmatrix} \begin{pmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{pmatrix} & & \\ & \begin{pmatrix} 0 & -\omega_2 \\ \omega_2 & 0 \end{pmatrix} & 0 \\ 0 & & A \end{pmatrix}$

where $\quad \sigma(A) \cap i\mathbb{R} = \emptyset$

Center Mnfds

Theorem (CM) Let f be a C^r-v.f on $\mathbb{R}^n$ s.t. $f(0)=0$.
Let $A = Df(0)$. Divide the spectrum of A into three parts $\sigma_s, \sigma_c, \sigma_u$ with

$$Re(\lambda) \begin{cases} <0 & \text{if } \lambda \in \sigma_s \\ =0 & \text{if } \lambda \in \sigma_c \\ >0 & \text{if } \lambda \in \sigma_u \end{cases}$$

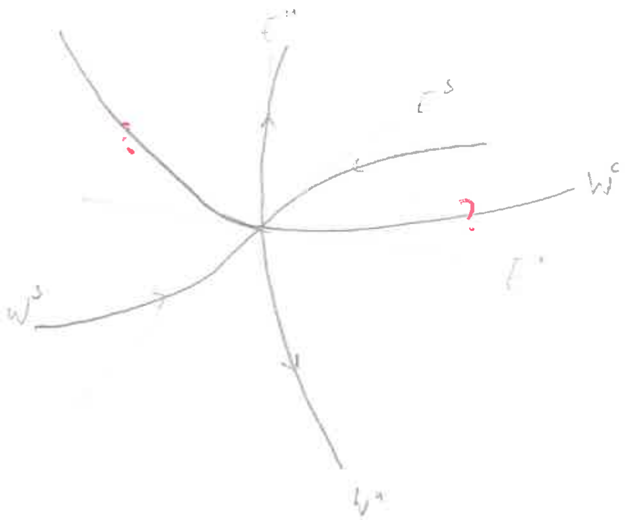
Let E^s, E^c, E^u be the generalized eigenspace of $\lambda \in \sigma_s, \sigma_c, \sigma_u$:

$$E^s = \bigoplus_{\lambda \in \sigma_s} E(\lambda), \quad E^c = \bigoplus_{\lambda \in \sigma_c} E(\lambda), \quad E^u = \bigoplus_{\lambda \in \sigma_u} E(\lambda).$$

Then (i) $\exists$ C^r stable and unstable inv. mnfds W^s, W^u tangent to E^s, E^u at 0, respectively;

(ii) $\exists$ C^{r-1} center mnfd W^c tangent to E^c at 0.

(iii) W^s, W^u are unique, but W^c need not be (Hw)

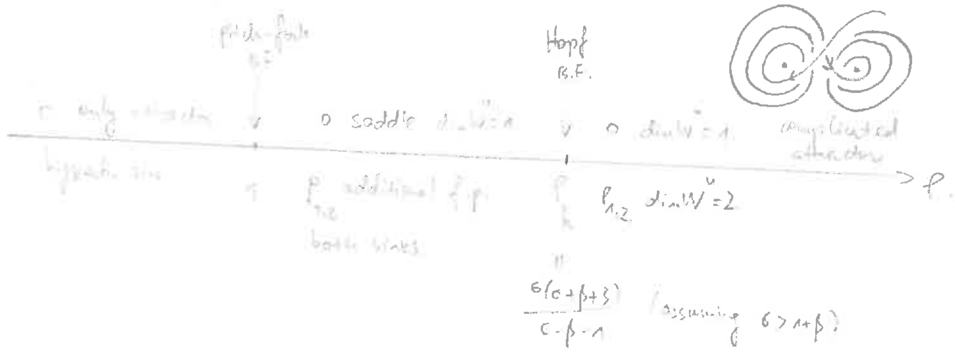


Example (E^c is not sufficient to derive dynamics on W^c)

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = -\beta z + xy \end{cases}$$

Lorenz system (E. Lorenz '63)

$x, y, z \in \mathbb{R}$, $\sigma, \rho, \beta > 0$ parameters.



We consider BF. at $(0,0,0)$ for $\rho = 1$

$$Df(0,0,0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix} \overset{\rho=1}{=} \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

$$\sigma(Df(0)) = \{ 0, \underbrace{-\sigma-1}_{<0}, \underbrace{-\beta}_{<0} \}$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} \sigma \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ eigenvectors.}$$

Let

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\sigma} & \frac{\sigma}{1+\sigma} & 0 \\ \frac{1}{1+\sigma} & -\frac{1}{1+\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

eigenbasis!

Then

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sigma-1 & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} -\frac{\sigma}{1+\sigma}(u+\sigma v)w \\ \frac{1}{1+\sigma}(u+\sigma v)w \\ (u+\sigma v)(u-v) \end{pmatrix}$$

diagonalized!

$$E^c_{(0,0,0)} = \text{the } u\text{-axis in } (u,v,w)\text{-space}$$
$$= \{ (u,0,0) : u \in \mathbb{R} \}.$$

E^c is not invariant, since

$$\begin{pmatrix} \dot u \\ \dot v \\ \dot w \end{pmatrix} \overset{v=w=0}{=\!=} \begin{pmatrix} 0 \\ 0 \\ u^2 \end{pmatrix}$$ — for $u \neq 0$, $\dot w \neq 0$, so w does NOT stay 0!

Set $\tilde w = w - \frac{u^2}{\beta}$. Then

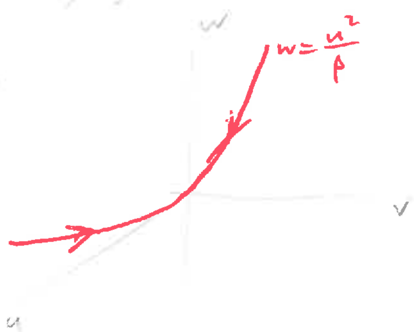
$$\begin{cases} \dot u = -\frac{\sigma}{1+\sigma}(u+\sigma v)\left(\tilde w + \frac{u^2}{\beta}\right) \\ \dot v = -(1+\sigma)v + \frac{1}{1+\sigma}(u+\sigma v)\left(\tilde w + \frac{u^2}{\beta}\right) \\ \dot{\tilde w} = -\beta \tilde w + (\sigma - 1)uv - \sigma v^2 + \frac{2\sigma}{\beta(1+\sigma)} u(u+\sigma v)\left(\tilde w + \frac{u^2}{\beta}\right) \end{cases}$$

$$\overset{v=\tilde w=0}{\Longrightarrow} \begin{cases} \dot u = -\frac{\sigma}{\beta(1+\sigma)} u^3 \\ \dot v = \frac{1}{\beta(1+\sigma)} u^3 \\ \dot{\tilde w} = \frac{2\sigma}{\beta^2(1+\sigma)} u^4 \end{cases}$$ ← a higher order than u^2!

$\{(u,v,\tilde w) \; v = \tilde w = 0\} = \{(u,v,w) \; v=0, \; w = \frac{u^2}{\beta}\}$ is not inv.

But it is invariant up to the 2nd order of u

In other words, $W^c_{(0)}$ looks appr (up to u^2) like:



and restricted flow on $W^c_{(0)}$ is given by:

$$\dot u = -\frac{\sigma}{\beta(1+\sigma)} u^3$$

i.e. on W^c, 0 is asym. stable.

We will see later that this is a pitch-fork BF from 0.

#

Supplementary : regular value theorem

• (topo. mfd) A topological space S is called a topological manifold, if S admits an open covering $\{U_\alpha\}$ s.t.

$$\forall_\alpha \quad \exists \text{ homeom } \varphi_\alpha : U_\alpha \to V_\alpha \quad \text{for } V_\alpha \subseteq \mathbb{R}^N \text{ open}$$

$(U_\alpha, \varphi_\alpha)$ is called a chart, $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ is called an atlas of S; N is the dimension of S.

• (continuous map) A map $f : S \to T$ between 2 topo. mfds S and T is called continuous, if

$$\forall_{p \in S} \quad \forall_{\substack{U_\alpha \in A_S \text{ with } p \in U_\alpha \\ V_\beta \in A_T \quad f(p) \in V_\beta}} \quad \psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \psi_\beta(f(U_\alpha))$$

is a continuous map,

where $A_S = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ resp. $A_T = \{(V_\beta, \psi_\beta)\}_\beta$ is the atlas of S resp. T.

$\bar{f} = \psi \circ f \circ \varphi^{-1}$ is called a local representation of f around p.

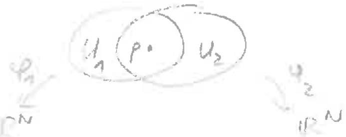
? differentiable map, if its local rep. is differentiable ?

$$\underbrace{\psi_1 \circ f \circ \varphi_1^{-1}}$$

$$\psi_2 \circ f \circ \varphi_2^{-1} = \boxed{\psi_2 \circ \psi_1^{-1}} \circ \underbrace{\psi_1 \circ f \circ \varphi_1^{-1}} \circ \boxed{\varphi_1 \circ \varphi_2^{-1}}$$

transition maps must be differentiable too !

• (C^r-compatible charts, C^r-Atlas)



Two charts (U_1, φ_1) and (U_2, φ_2) in a topo. mnfd M (for which $U_1 \cap U_2 \neq \emptyset$) are called <u>$C^r$-compatible</u>, if the transition map $\varphi_1 \circ \varphi_2^{-1}$ is C^r. An Atlas A of M is called <u>C^r-Atlas</u>, if every two charts from A are C^r-compatible. A is called <u>saturated</u>, if it contains all charts that are C^r compatible with charts of A. A saturated atlas is called a <u>C^r-differentiable structure</u> of M.

• (C^r-Mnfd, C^r-map) A topo. mnfd with a C^r-diff structure is called a <u>C^r-differentiable mnfd</u>. A C^∞-mnfd is also called a <u>smooth mnfd</u>.

A continuous map $f: M \to N$ between two C^r-mnfds M, N is called <u>C^r-differentiable</u>, if its local representation $\bar{f} = \psi \circ f \circ \varphi^{-1}$ is C^r-differentiable for every $\varphi \in A_M$, $\psi \in A_N$.

• (Submnfd) Let M be a C^r-mnfd of dimension N and $N \subset M$ be a subset. N is called an n-dim'l <u>C^r-submnfd</u> of M, if

$$\forall_{x \in N} \; \exists_{\substack{(U, \varphi) \in A_M \\ x \in U}} \text{ s.t. } \varphi(N \cap U) = \varphi(U) \cap \mathbb{R}^n, \text{ where}$$

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_{n+1} = \ldots = x_N = 0\}$$

dim m dim n

• (regular point, regular value) Let $f: M \to N$ be a C^1-map. A point $p \in M$ is called a <u>regular point</u> of f, if the linear operator

$$D\left(\psi_\beta \circ f \circ \varphi_\alpha^{-1}\right)_{\varphi_\alpha(p)} : \mathbb{R}^m \to \mathbb{R}^n$$

for some charts $\varphi_\alpha \in \mathcal{A}_M$, $\psi_\beta \in \mathcal{A}_N$ is surjective. Otherwise, p is called <u>critical</u>.

⌈ <u>Note</u>: if $m < n$, then every point is critical;
if $m \geq n$, then p is regular $\Leftrightarrow$ rank $D(\psi_\beta \circ f \circ \varphi_\alpha^{-1}) = n$ ⌋

A value $q \in N$ is called <u>regular value</u> of f, if every its inverse image is regular. Otherwise, q is called <u>critical</u>.
⌈ If $q \notin f(M)$ then q is by definition regular. ⌋

<u>Theorem</u> (<u>regular value theorem</u>) Let $f: M \to N$ be a C^1-map and $q \in N$ be a regular value. Then $f^{-1}(q)$ is either empty or is a submfd of N with dimension $=$ dim $M -$ dim N

<u>proof</u>: dim $M = m+n$, dim $N = n$.

$$\begin{tikzcd}
M \supseteq U \arrow[r, "f"] & V \subseteq N \\
p \arrow[r] \arrow[d, "\varphi"'] & q \arrow[d, "\psi"] \\
\mathbb{R}^{m+n} \arrow[r, "\bar{f}"] & \mathbb{R}^n \\
0 \arrow[r] & 0
\end{tikzcd}$$

q is regular $\Rightarrow$ rank $D\bar{f}(0) = n$.

Up to reordering of coordinates in $\mathbb{R}^{m+n}$ and $\mathbb{R}^n$, we have

$$D\bar{f}(0) = \begin{array}{cc} \left(\begin{array}{c|c} A & B \end{array} \right) \\ {\scriptstyle n\times n} \quad {\scriptstyle n\times m} \end{array} \quad \text{where } A \text{ is invertible}$$

Define $h : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^{m+n}$

$$x = (x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{n+m}) \longmapsto (\underbrace{\bar{f}(x)}_{\mathbb{R}^n}, x_{n+1}, \dots, x_{n+m}).$$

$$\Rightarrow \begin{cases} h(0) = (\underset{\underset{0}{\parallel}}{\bar{f}(0)}, 0) = 0 \\ Dh(0) = \begin{pmatrix} A & B \\ 0 & Id \end{pmatrix} \Rightarrow Dh(0) \text{ is invertible} \end{cases}$$

$\overset{\text{I.F.T.}}{\Longrightarrow}$ h is a local diffeom. around 0.

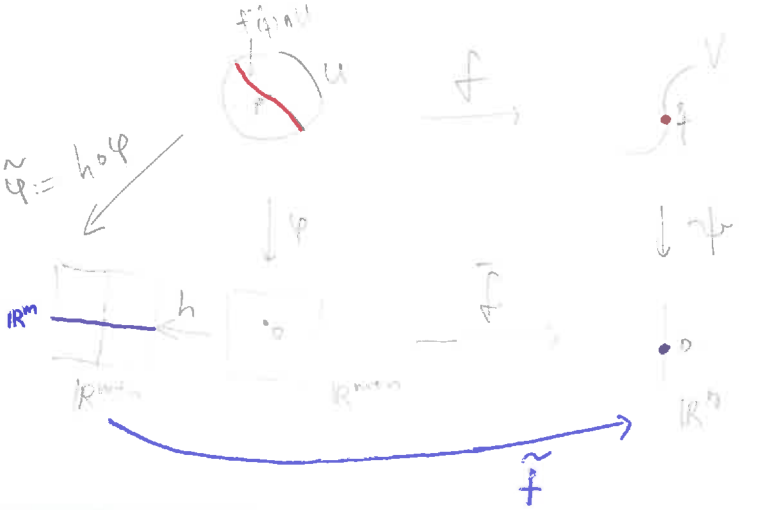
Consider

$$\begin{tikzcd} \mathbb{R}^{m+n} \arrow[r, "h^{-1}"] \arrow[rr, bend right, "\tilde{f}"'] & \mathbb{R}^{m+n} \arrow[r, "\bar{f}"] & \mathbb{R}^n \end{tikzcd},$$

where

$$(\underbrace{y_1, y_2, \dots, y_n}_{=\bar{f}(x)}, x_{n+1}, \dots, x_{n+m}) \xrightarrow{h^{-1}} (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}) \xrightarrow{\bar{f}} (y_1, \dots, y_n),$$

i.e. $\tilde{f} : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is a projection.



i.e. $\forall p \in f^{-1}(q)$ $\exists (U, \underset{\underset{h\circ\varphi}{\parallel}}{\tilde{\varphi}})$ s.t.

$$\tilde{\varphi}(f^{-1}(q) \cap U) = \tilde{\varphi}(U) \cap \mathbb{R}^m$$

$\Rightarrow$ $f^{-1}(q)$ is a submfd of dim m. $\blacksquare$

Calculation of Center Manifolds

$$(*) \qquad \dot{x} = f(x) \quad , \quad x \in \mathbb{R}^N$$

where $f(0)=0$ und $J = Df(0)$.

$$\sigma(J) = \sigma_u \sqcup \sigma_c \sqcup \sigma_s$$

$$\mathbb{R}^N = E^u \oplus E^c \oplus E^s$$

$W^u \quad W^c \quad W^s$ tangent at 0

$\mathbb{R}^N \xrightarrow{J} \mathbb{R}^N$ Eigenbasis $E^u \times E^c \times E^s \longrightarrow E^u \times E^c \times E^s$ (A, B, C)

$$J \xrightarrow{\text{eigenbasis}} \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

For simplicity, we assume $\sigma_u = \phi \quad \Rightarrow \quad J \cong \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$

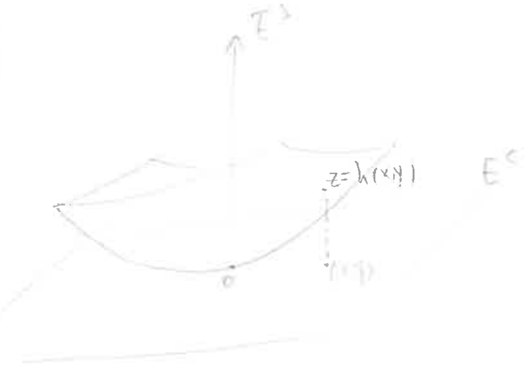
$n := \dim E^c$, $\quad m = \dim E^s$

$$(*) \quad \dot{x} = f(x) \, , \; x \in \mathbb{R}^N \xrightarrow[\text{of } J]{\text{eigenbasis}} \begin{cases} (*1) \quad \dot{x} = Bx + f(x,y) \\ (*2) \quad \dot{y} = Cy + g(x,y) \end{cases} \qquad (x,y) \in E^c \times E^s$$

where

$$f(0,0) = g(0,0) = 0$$

$$f_x(0,0) = f_y(0,0) = g_x(0,0) = g_y(0,0) = 0$$



$W_0^c = \{(x,y) \in E^c \times E^s : y = h(x), \ |x| \ll 1,$
$h(0)=0, \ Dh(0)=0\}$

$h = h(x)$ satisfies

$$\left.\begin{array}{l} \dot{x} = Bx + f(x,y) \\ \dot{y} = Cy + g(x,y) \\ y = h(x) \end{array}\right\} \Rightarrow \underset{\substack{\| \\ Ch(x)+g(x,h(x))}}{\dot{y}} = Dh(x)\,\dot{x} = Dh(x)\big(Bx + f(x,h(x))\big)$$

$$\Rightarrow (N): \ \boxed{Dh(x)\,(Bx + f(x,h(x))) - Ch(x) - g(x,h(x)) = 0}$$

$h = h(x)$ satisfies the nonlinear partial diff eqn (N) subject to the boundary conditions $h(0)=0$, $Dh(0)=0$.

- Approximative solution of h:

$$h(x) = \underbrace{\text{Taylor expansion around } 0}_{\hat{h}(x)} + O(|x|^p).$$

- Approximative dynamic on W^c.

$$\dot{x} = Bx + f(x, \hat{h}(x))$$

__Satz__ (Henry '81, Carr '81) If 0 is locally asym. stable (resp. unstable) for $\dot{x} = Bx + f(x, h(x))$, then 0 is also locally asym stable (resp. unstable) for

$$\begin{cases} \dot{x} = Bx + f(x,y) \\ \dot{y} = Cy + g(x,y) \end{cases}$$

__Satz__ (Henry-Carr '81) If a function $\phi(x)$ with $\phi(0)=0$, $D\phi(0)=0$ can be found such that $N(\phi(x)) = O(|x|^p)$ for some $p>1$ and $|x| \to 0$, then $h(x) = \phi(x) + O(|x|^p)$ as $|x| \to 0$.

__Bsp__ (Lorenz revisited, part I)

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = -\beta z + xy \end{cases} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3, \quad \sigma, \rho, \beta > 0 \text{ parameters}$$

For $\rho = 1$, consider BF around $(0,0,0)$

$$J = Df(0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix} \qquad \sigma(J) = \{0, -\sigma-1, -\beta\}$$

Using eigenbasis of J:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{1+\sigma} & \frac{\sigma}{1+\sigma} & 0 \\ \frac{1}{1+\sigma} & -\frac{1}{1+\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{= P^{-1}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad P^{-1} J P = \Lambda$$

$$\Rightarrow \begin{cases} \dot u = \overset{B}{0} \underbrace{\left(-\frac{\sigma}{1+\sigma}(u+\sigma v) w\right)}_{f} \\ \begin{pmatrix} \dot v \\ \dot w \end{pmatrix} = \underbrace{\begin{pmatrix} -\sigma-1 & 0 \\ 0 & -\beta \end{pmatrix}}_{C} \begin{pmatrix} v \\ w \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{1}{1+\sigma}(u+\sigma v)w \\ (u+\sigma v)(u-v) \end{pmatrix}}_{g} \end{cases}$$

$\mathbb{R}^3 = E^c \times E^s = u\text{-axis} \times (v,w)\text{-plane}$

$W_0^c = \{ (u,v,w) \in \mathbb{R}^3 : \; v = h_1(u), \; w = h_2(u), \; h_1(0) = h_2(0) = 0, \; h_1'(0) = h_2'(0) = 0 \}$

(N): $\quad Dh(x)\,(Bx + f(x,h(x))) - C h(x) - g(x,h(x)) = 0$

$$\Rightarrow \begin{pmatrix} h_1'(u) \\ h_2'(u) \end{pmatrix} \underbrace{f(u,h_1,h_2)}_{-\frac{\sigma}{1+\sigma}(u+\sigma h_1)h_2} - \begin{pmatrix} -\sigma-1 & 0 \\ 0 & -\beta \end{pmatrix} \begin{pmatrix} h_1(u) \\ h_2(u) \end{pmatrix} - \underbrace{g(u,h_1,h_2)}_{\begin{pmatrix} \frac{1}{1+\sigma}(u+\sigma h_1)h_2 \\ (u+\sigma h_1)(u-h_1) \end{pmatrix}} = 0$$

$$\Rightarrow \begin{cases} -\frac{\sigma}{1+\sigma}(u+\sigma h_1)h_2\, h_1' + (\sigma+1)h_1 - \frac{1}{1+\sigma}(u+\sigma h_1)h_2 = 0 \\ -\frac{\sigma}{1+\sigma}(u+\sigma h_1)h_2\, h_2' + \beta h_2 - (u+\sigma h_1)(u-h_1) = 0 \end{cases}$$

multip. $(1+\sigma)$ both sides, let $h_1(u) = a_1 u^2 + b_1 u^3 + c_1 u^4 + O(u^5)$

$h_2(u) = a_2 u^2 + b_2 u^3 + c_2 u^4 + O(u^5)$

$$\begin{cases} -\sigma\left(u+\sigma a_1u^2+\sigma b_1u^3+\sigma c_1u^4+\dots\right)\left(a_2u^2+b_2u^3+c_2u^4+\dots\right)\left(2a_1u+3b_1u^2+4c_1u^3+\dots\right) \\ +(\sigma+1)^2\left(a_1u^2+b_1u^3+c_1u^4+\dots\right)-\left(u+\sigma a_1u^2+\sigma b_1u^3+\sigma c_1u^4+\dots\right)\left(a_2u^2+b_2u^3+c_2u^4+\dots\right) \\ \qquad = 0 \\ u^2\text{- coeff}: \quad a_1(\sigma+1)^2 = 0 \quad \Rightarrow \quad a_1 = 0 \\ u^3\text{- coeff}: \quad b_1(\sigma+1)^2 - a_2 = 0 \quad \Rightarrow \quad b_1 = a_2/(\sigma+1)^2 = \frac{1}{\beta(\sigma+1)^2} \\ -\sigma\left(u+\sigma a_1u^2+\sigma b_1u^3+\sigma c_1u^4+\dots\right)\left(a_2u^2+b_2u^3+c_2u^4+\dots\right)\left(2a_2u+3b_2u^2+4c_2u^3+\dots\right) \\ +(1+\sigma)\beta\left(a_2u^2+b_2u^3+c_2u^4+\dots\right)-(1+\sigma)\left(u+\sigma a_1u^2+\sigma b_1u^3+\dots\right)\left(u-a_1u^2-b_1u^3-\dots\right)=0 \\ u^2\text{- coeff}: \quad (1+\sigma)\beta a_2 - (1+\sigma) \quad \Rightarrow \quad a_2 = \frac{1}{\beta} \\ u^3\text{- coeff}: \quad (1+\sigma)\beta b_2 - (1+\sigma)(\sigma a_1 - a_1) \overset{a_1=0}{\Longrightarrow} b_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} h_1(u) = \dfrac{1}{\beta(\sigma+1)^2}u^3 + O(u^4) \\ h_2(u) = \dfrac{1}{\beta}u^2 + O(u^4) \end{cases} \Rightarrow W^c \doteq \{(u, h_1, h_2) : \dots\}$$

On W^c:

$$\dot{u} = -\frac{\sigma}{1+\sigma}(u+\sigma v)w$$

$$= -\frac{\sigma}{1+\sigma}(u+\sigma h_1)h_2$$

$$\doteq -\frac{\sigma}{1+\sigma}\left(u+\frac{\sigma}{\beta(\sigma+1)^2}u^3\right)\cdot\frac{u^2}{\beta} = \underbrace{-\frac{\sigma}{(1+\sigma)\beta}}_{<0}u^3 - \frac{\sigma^2}{(1+\sigma)^3\beta^2}u^5$$

#

Bsp. (Approx. using Taylor expansion may fail if W^c is not analytic around 0)

$$\begin{cases} \dot{x} = -x^3 \\ \dot{y} = -y + x^2 \end{cases} \quad \text{around } (0,0).$$

(Hw) Show that $W^c = \{(x, y(x)) : x \in \mathbb{R}\}$ with

(S) $$y(x) = e^{-\frac{1}{2x^2}} \left\{ C + \frac{1}{2} \ln\left(\frac{1}{2x^2}\right) + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^n n \, n! \, x^{2n}} \right\}$$

for $C \in \mathbb{R}$ is a family of center manifolds for the system, which is not analytic around 0.

Using $h(x) = ax^2 + bx^3 + cx^4 + O(x^5)$:

$$h'(x) \cdot (-x^3) + h(x) - x^2 = 0$$

$\Rightarrow$ $$-x^3(2ax + 3bx^2 + 4cx^3 + \dots) + ax^2 + bx^3 + cx^4 \dots - x^2 = 0$$

$\Rightarrow$ x^2-coeff: $a - 1 = 0 \Rightarrow a = 1$

x^3-coeff: $b = 0$

x^4-coeff: $-2a + c = 0 \Rightarrow c = 2$

$\Rightarrow$ $h(x) = x^2 + 2x^4 + O(x^5)$ is an approx of $\underline{\underline{a}}$ center mnfd but not of those by (S).

#

Calculation of W^c for parametrised systems

Consider $\dot{x} = F_\mu(x)$, $x \in \mathbb{R}^N$, $\mu \in \mathbb{R}^k$ parameter.

Assume $F_\mu(0) = 0$ $\forall_\mu$ in a nbhd of $\mu_0 \in \mathbb{R}^k$

Set $J := J_{\mu_0} = DF_{\mu_0}(0)$. Assume $\sigma(J) \cap i\mathbb{R} \neq \phi$.

We derive W^c_0 for $\mu = \mu_0$ as follows:

- using eigenbasis write $J = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, where $Re(\lambda) < 0$ $\forall_{\lambda \in \sigma(C)}$, $Re(\lambda) = 0$ $\forall_{\lambda \in \sigma(B)}$

(we assume here $\sigma(J) = \sigma_s \sqcup \sigma_c \sqcup \sigma_u$, with $\sigma_u = \phi$), and the system can be rewritten as (w.r.t. eigenbasis)

$$\begin{cases} \dot{x} = B_\mu x + f_\mu(x,y) \\ \dot{y} = C_\mu y + g_\mu(x,y) \end{cases}, \quad \text{where } B_{\mu=\mu_0} = B,\ C_{\mu=\mu_0} = C.$$

- consider

$$\begin{cases} \dot{x} = B_\mu x + f_\mu(x,y) \\ \dot{\mu} = 0 \\ \dot{y} = C_\mu y + g_\mu(x,y) \end{cases} \Leftrightarrow \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{\mu} \end{pmatrix} = \overbrace{\boxed{\begin{pmatrix} B_\mu & 0 \\ 0 & 0 \end{pmatrix}}}^{B} \begin{pmatrix} x \\ \mu \end{pmatrix} + \begin{pmatrix} f_\mu(x,y) \\ 0 \end{pmatrix} \leftarrow f \\ \dot{y} = \underbrace{\boxed{C_\mu}}_{C} y + g_\mu(x,y) \leftarrow g \end{cases}$$

One can apply "(N): $Dh(x)(Bx + f(x,h(x))) - C \cdot h(x) - g(x,h(x)) = 0$"
where $x = \begin{pmatrix} x \\ \mu \end{pmatrix}$ and $h = h(x,\mu) = ax^2 + b\mu^2 + cx\mu + O(|x,\mu|^3)$, to determine
$W^c = \{(x,y,\mu) \in \mathbb{R}^N \times \mathbb{R}^k : y = h(x,\mu)\}$.

Example 2.2.2. (continued.) Consider the Lorenz system
(part II)

near $\rho = 1$:

$$(L)\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = -\beta z + xy \end{cases} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3, \quad \sigma, \beta > 0, \quad \rho \text{ near } 1.$$

Using the same eigenbasis transformation:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\sigma} & \frac{\sigma}{1+\sigma} & 0 \\ \frac{1}{1+\sigma} & \frac{-1}{1+\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{i.e.} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

we have (L):

$$\begin{cases} \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \frac{\sigma(\rho-1)}{1+\sigma} & \frac{\sigma^2(\rho-1)}{1+\sigma} & 0 \\ -\frac{\rho-1}{1+\sigma} & -\frac{\sigma(\sigma+\rho)}{1+\sigma} - 1 & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} -\frac{\sigma}{1+\sigma}(u+\sigma v)w \\ \frac{1}{1+\sigma}(u+\sigma v)w \\ (u+\sigma v)(u-v) \end{pmatrix} \\ \text{together with} \\ \dot{\rho} = 0 \end{cases}$$

At $(u,v,w,\rho) = (0,0,0,1)$, the above system has center manifold

$$E^c = \text{"}u\text{-axis"} \times \text{"}\rho\text{-axis"} \simeq \mathbb{R}^2 \subseteq \mathbb{R}^4$$

thus $W^c = \{(u,v,w,\rho) : v = h_1(u,\rho),\ w = h_2(u,\rho)\}$ $\quad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$.

$$\dot u = \frac{\sigma\tilde\rho}{1+\sigma}u + \frac{\sigma^2\tilde\rho}{1+\sigma}v - \frac{\sigma}{1+\sigma}(u+\sigma v)w$$

$$\left\{\begin{aligned}
\begin{pmatrix}\dot u\\ \dot{\tilde\rho}\end{pmatrix} &= \begin{pmatrix}\frac{\sigma\tilde\rho}{1+\sigma} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}u\\ \tilde\rho\end{pmatrix} + \begin{pmatrix}\frac{\sigma^2\tilde\rho v}{1+\sigma} - \frac{\sigma}{1+\sigma}(u+\sigma v)w\\ 0\end{pmatrix}\\
\begin{pmatrix}\dot v\\ \dot w\end{pmatrix} &= \begin{pmatrix}-\left(\sigma+1+\frac{\sigma}{1+\sigma}\tilde\rho\right) & 0\\ 0 & -\beta\end{pmatrix}\begin{pmatrix}v\\ w\end{pmatrix} + \begin{pmatrix}-\frac{\tilde\rho u}{1+\sigma} + \frac{1}{1+\sigma}(u+\sigma v)w\\ (u+\sigma v)(u-v)\end{pmatrix}.
\end{aligned}\right.$$

$$h = \begin{pmatrix}h_1\\ h_2\end{pmatrix} = \begin{pmatrix}a_1u^2+b_1u\tilde\rho+c_1\tilde\rho^2+d_1u^3+\dots\\ a_2u^2+b_2u\tilde\rho+c_2\tilde\rho^2+d_2u^3+\dots\end{pmatrix}$$

$$\begin{pmatrix}2a_1u+b_1\tilde\rho+3d_1u^2 & b_1u+2c_1\tilde\rho\\ 2a_2u+b_2\tilde\rho+3d_2u^2 & b_2u+2c_2\tilde\rho\end{pmatrix}\left[\begin{pmatrix}\frac{\sigma\tilde\rho}{1+\sigma} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}u\\ \tilde\rho\end{pmatrix} + \begin{pmatrix}\frac{\sigma^2\tilde\rho h_1}{1+\sigma} - \frac{\sigma}{1+\sigma}(u+\sigma h_1)h_2\\ 0\end{pmatrix}\right]$$

$$+\begin{pmatrix}\sigma+1+\frac{\sigma}{1+\sigma}\tilde\rho & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}h_1\\ h_2\end{pmatrix} - \begin{pmatrix}-\frac{\tilde\rho u}{1+\sigma} + \frac{1}{1+\sigma}(u+\sigma h_1)h_2\\ (u+\sigma h_1)(u-h_1)\end{pmatrix} = 0.$$

$$\left(2a_1u+b_1\tilde{\rho}+3d_1u^2+\dots\right)\Bigg[\frac{\sigma}{1+\sigma}\tilde{\rho}u+\frac{\sigma^2}{1+\sigma}\left(a_1\tilde{\rho}u^2+b_1u\tilde{\rho}^2+c_1\tilde{\rho}^3+d_1u^3\tilde{\rho}+\dots\right)$$

$$-\frac{\sigma}{1+\sigma}\left(u+a_1\sigma u^2+b_1\sigma u\tilde{\rho}+c_1\sigma\tilde{\rho}^2+d_1\sigma u^3+\dots\right)\left(a_2u^2+b_2u\tilde{\rho}+c_2\tilde{\rho}^2+d_2u^3+\dots\right)\Bigg]$$

$$+\left(\sigma+1+\frac{\sigma}{1+\sigma}\tilde{\rho}\right)\left(a_1u^2+b_1u\tilde{\rho}+c_1\tilde{\rho}^2+d_1u^3+\dots\right)+\frac{\tilde{\rho}u}{1+\sigma}-\frac{1}{1+\sigma}\left(u+a_1\sigma u^2+b_1\sigma u\tilde{\rho}+c_1\sigma\tilde{\rho}^2+\right.$$

$$\left.d_1\sigma u^3+\dots\right)\left(a_2u^2+b_2u\tilde{\rho}+c_2\tilde{\rho}^2+d_2u^3+\dots\right)=0.$$

u^2-

$$(\sigma+1)a_1=0$$

$u\tilde{\rho}-$

$$(\sigma+1)b_1+\frac{1}{1+\sigma}=0 \Rightarrow b_1=-\frac{1}{(1+\sigma)^2}$$

$\tilde{\rho}^2-$

$$(\sigma+1)c_1=0 \Rightarrow c_1=0$$

u^3-

$$-\frac{\sigma}{1+\sigma}a_2+(\sigma+1)d_1-\frac{1}{1+\sigma}a_2=0.$$

$$\Rightarrow -a_2+(\sigma+1)d_1=0 \Rightarrow d_1=\frac{a_2}{\sigma+1}=\frac{1}{\beta(\sigma+1)}$$

$$\left(2a_2u+b_2\tilde{\rho}+3d_2u^2+\dots\right)\Big[\frac{\sigma}{1+\sigma}\tilde{\rho}u+\frac{\sigma^2}{1+\sigma}\left(a_1\tilde{\rho}u^2+b_1u\tilde{\rho}^2+c_1\tilde{\rho}^3+d_1u^3\tilde{\rho}+\dots\right)$$

$$-\frac{\sigma}{1+\sigma}\left(u+a_1\sigma u^2+b_1\sigma u\tilde{\rho}+c_1\sigma\tilde{\rho}^2+d_1\sigma u^3+\dots\right)\left(a_2u^2+b_2u\tilde{\rho}+c_2\tilde{\rho}^2+d_2u^3+\dots\right)\Big]$$

$$+\beta\left(a_2u^2+b_2u\tilde{\rho}+c_2\tilde{\rho}^2+d_2u^3+\dots\right)-\left(u+\sigma a_1u^2+\sigma b_1u\tilde{\rho}+\sigma c_1\tilde{\rho}^2+\sigma d_1u^3+\dots\right)\cdot\left(u-a_1u^2-b_1u\tilde{\rho}-c_1\tilde{\rho}^2-d_1u^3-\dots\right)$$

$=0.$

u^2 – $\quad \beta a_2 - 1 = 0. \quad \Rightarrow \quad a_2 = \frac{1}{\beta}$

$u\tilde{\rho}$ – $\quad \beta b_2 = 0 \quad \Rightarrow \quad b_2 = 0$

$\tilde{\rho}^2$ – $\quad \beta c_2 = 0 \quad \rightarrow \quad c_2 = 0$

u^3 – $\quad -\frac{\sigma}{1+\sigma}a_2 + \beta d_2 + a_1 - \sigma a_1 = 0$

$$d_2 = \frac{\sigma a_2}{(1+\sigma)\beta} = \frac{\sigma}{(1+\sigma)\beta^2}$$

$$h_1 = -\frac{1}{(1+\sigma)^2}u\tilde{\rho} + \frac{1}{\beta(\sigma+1)}u^3 + \dots$$

$$h_2 = \frac{1}{\beta}u^2 + \frac{\sigma}{(1+\sigma)\beta^2}u^3 + \dots$$

$$\dot{u} = \frac{\sigma\tilde{\rho}}{1+\sigma}u + \frac{\sigma^2\tilde{\rho}}{1+\sigma}h_1 - \frac{\sigma}{1+\sigma}(u+\sigma h_1)h_2$$

$$= \frac{\sigma\tilde{\rho}}{1+\sigma}u + \frac{\sigma^2\tilde{\rho}}{1+\sigma}\left(-\frac{1}{(1+\sigma)^2}u\tilde{\rho} + \frac{1}{\beta(\sigma+1)}u^3 + \dots\right)$$

$$- \frac{\sigma}{1+\sigma}\left(u - \frac{\sigma}{(1+\sigma)^2}u\tilde{\rho} + \frac{\sigma}{\beta(1+\sigma)}u^3 + \dots\right)\left(\frac{1}{\beta}u^2 + \frac{\sigma}{(1+\sigma)\beta^2}u^3 + \dots\right)$$

$$= \left(\frac{\sigma\tilde{\rho}}{1+\sigma} - \frac{\sigma^2\tilde{\rho}^2}{(1+\sigma)^3}\right)u + \frac{\sigma^2\tilde{\rho}}{\beta(1+\sigma)^2}u^3 + \dots$$

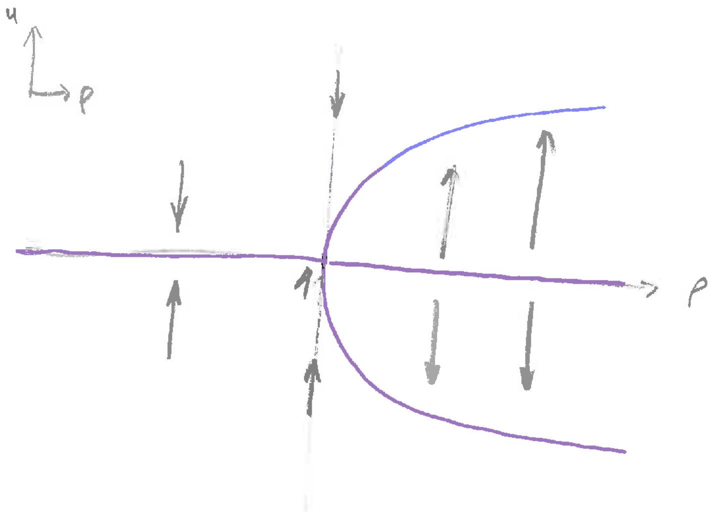
$$- \frac{\sigma}{\beta(1+\sigma)}u^3 + \dots$$

$$= \frac{\sigma\tilde{\rho}}{1+\sigma}\left(1 - \frac{\sigma\tilde{\rho}}{(1+\sigma)^2}\right)u + \underbrace{\left(\frac{\sigma^2\tilde{\rho}}{\beta(1+\sigma)^2} - \frac{\sigma}{\beta(1+\sigma)}\right)}_{= \underbrace{\frac{\sigma}{\beta(1+\sigma)^2}\left(\sigma\tilde{\rho} - (1+\sigma)\right)}_{<0.}} u^3 + \dots$$

$\hat{\rho} = 0$

Essentially, on W^c, we have

$$\dot{u} = (\rho-1)u - u^3 \qquad (\text{compare with Example } 2.1.3)$$



when $\rho<1$, $u=0$ is the only FP, and it is stable,

when $\rho>1$, $u=\pm\sqrt{\rho-1}$ are additional FPs, and they are stable, $u=0$ becomes unstable.

when $\rho=1$, $\dot{u}=-u^3$, $\Rightarrow$ $u=0$ is the only FP, and is weakly stable

(this coincides with our calculation in Example 2.2.2 (part I)).

#

2.3. Normal forms (simplified forms of flows)

Consider $\dot{x} = f(x)$, $x \in \mathbb{R}^N$, with $f(0)=0$.

Depending on eigenvalues of $Df(0)$, we consider 2 cases:

Case A: non-resonance

Case B: resonance

Def 2.3.1 Let $\lambda_1, \dots, \lambda_N \in \mathbb{C}$. If there exist $a_1, \dots, a_N \in \mathbb{N} \cup \{0\}$ with $\sum_{j=1}^{N} a_j \geq 2$ such that for some λ_i:

$$\lambda_i = \sum_{j=1}^{N} a_j \lambda_j ,$$

then $\lambda_1, \dots, \lambda_N$ are said to be resonant of order $|a| := \sum_{j=1}^{N} a_j$

Classical examples of resonance:

(I) zero eigenvalue: $\lambda_i = \lambda_i + 0 = \lambda_i + 2 \cdot 0 = \cdots$ any order ≥ 2.

(II) pure imaginary eigenvalues $\pm i\beta$: $\lambda_i = \lambda_i + (i\beta + (-i\beta)) = \lambda_i + 2 \cdot (i\beta + (-i\beta)) = \cdots$ any odd order ≥ 3.

<u>Case A</u> $Df(0)$ has nonresonant eigenvalues $\lambda_1, \dots, \lambda_N$.

Using eigenbasis of $Df(0)$, we write $\dot{x} = f(x)$ as

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + g_1(x_1, \dots, x_N) \\ \dot{x}_2 = \lambda_2 x_2 + g_2(x_1, \dots, x_N) \\ \vdots \\ \dot{x}_N = \lambda_N x_N + g_N(x_1, \dots, x_N) \end{cases} \qquad \text{or} \qquad \dot{x} = \Lambda x + g(x),$$

where $g(0)=0$, $Dg(0)=0$, i.e. the smallest degree of a nonzero derivative of any g_i is at least 2.

Let k = the smallest degree of nonzero derivative of all $g_1, \dots, g_N$ $(k \geq 2)$

We want to find a coordinate transformation (nonlinear)

$$x = h(y) = y + P(y), \quad \text{where } P \text{ is a polynomial of degree } k,$$

such that the resulting system $\quad \dot{x} = (I + DP(y))\dot{y}$

$$(*) \qquad \dot{y} = (I + DP(y))^{-1}(\Lambda \cdot y + \Lambda \cdot P(y) + g(y + P(y)))$$

$$:= \Lambda \cdot y + \eta(y)$$

satisfies

$$k' := \text{the smallest degree of nonzero derivative of all } \eta_1, \dots, \eta_N \geq k+1$$

Considering $(*)$ but only keeping terms of degree $\leq k$:

$$(I+DP)(I-DP) = I - \cancel{DP} + \cancel{DP} - DPDP = I - \underbrace{DPDP}_{\text{lowest degree} \,=\, k-1+k-1 \,=\, 2k-2 \,\geq\, k}$$

$(k \geq 2)$

$$\Rightarrow \quad (I+DP)^{-1} = I - DP + O(y^k)$$

$$\dot{y} = \underbrace{(I+DP(y))^{-1}}_{\substack{\| \\ I-DP(y)+O(y^k)}} \underbrace{(\Lambda y + \Lambda P(y) + g(y+P(y)))}_{O(y)}$$

$$= (I-DP(y))(\Lambda y + \Lambda P(y) + g(y+P(y))) + O(y^{k+1})$$

$$\doteq \Lambda y + \Lambda P(y) + \underbrace{g(y+P(y))}_{\substack{\| \\ g^k(y)+O(y^{k+1})}} - DP(y)\Lambda y \boxed{- DP(y)\Lambda P(y) - DP(y) g(y+P(y))}$$

smallest degree: $1 \quad k \quad k \;(1 \;\; k) \quad k-1 \quad 1 \quad | \quad k-1 \quad k \quad k-1 \quad k$

degree higher than k

g^k: leading term in g for the degree k

$$\dot{y} = \Lambda y + \underbrace{\Lambda P(y) + g^k(y) - DP(y)\cdot\Lambda\cdot y}_{\overset{\text{let}}{=}\,0} + O(y^{k'}), \quad \text{s.t. } k' \geq k+1$$

That is, we seek for $P = \begin{pmatrix} P_1 \\ \vdots \\ P_N \end{pmatrix}$ satisfying

$$(*_p) \qquad F(P_i) := \lambda_i P_i(y) - \sum_{j=1}^{N} \frac{\partial P_i}{\partial y_j} \lambda_j y_j = -g_i^k(y), \qquad i=1,\dots,N$$

where P_i is a polynomial of degree k.

$(*_p)$ is linear in coefficients of P_i $\Rightarrow$ only need to consider monomials P_i

Let $P_i = y_1^{a_1} y_2^{a_2} \cdots y_N^{a_N}$ with $\sum_{j=1}^{N} a_j = k$, a monomial of degree k

Then $\frac{\partial P_i}{\partial y_j}\cdot y_j = a_j P_i \quad \Rightarrow \quad F(P_i) = \lambda_i P_i - \sum_{j=1}^{N} a_j \lambda_j P_i = (\lambda_i - \sum_{j=1}^{N} a_j \lambda_j) P_i$

i.e. monomial $(a_1,\dots,a_N)$ is an eigenvector of operator F for eigenvalue $(\lambda_i - \sum_{j=1}^{N} a_j\lambda_j)$.

$\Rightarrow (*_p)$ can be solved, if $\underbrace{\lambda_i - \sum_{j=1}^{N} a_j \lambda_j \neq 0 \quad \forall i \quad \forall (a_1,\dots,a_N) \text{ with } \sum_{j=1}^{N} a_j = k, \; a_j \geq 0,}_{\text{satisfied due to nonresonance assumption.}}$

Conclusion: if the eigenvalues $\lambda_1, \dots, \lambda_N$ of $Df(0)$ are non resonant of all order $k \geq 2$, then the system $\dot{x} = f(x)$, $f(0) = 0$ can be linearized (using polynomial change of coordinates) to any desired algebraic order.

---

case B: $Df(0)$ has resonant eigenvalues $\lambda_1, \dots, \lambda_N$

↑ more interesting case for bifurcation problems.

Assume $\lambda_1, \dots, \lambda_N$ are resonant of order k with

$$\lambda_i - \sum_{j=1}^{N} a_j \lambda_j = 0 \quad \text{for some } \lambda_i \text{ and } \sum_{j=1}^{N} a_j = k, \; a_j \geq 0,$$

then in $(*_p)$, LHS=0 for monomial $(a_1, \dots, a_N)$, thus if $g_i^k = g_i^k(y)$ has nonzero monomial $(a_1, \dots, a_N)$-term, then this term can NOT be removed by any change of coordinates; "essential term" or "resonance term"

Goal: to find change of coordinates, so only essential terms remain in the form of f, and this form is called the normal form of f.

To state the theorem of normal forms, a couple of notations:

Let H_k be the linear space of vec. fields whose coefficients are homogeneous polyn. of degree k, i.e.

$$H_k = \left\{ \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{pmatrix} : \begin{array}{l} P_i \text{ is a homog. polyn. of degree } k \text{ in } (x_1, \dots, x_N) \\ \text{i.e. } P_i \text{ is a lin. comb. of monomials of deg. } k. \end{array} \right\}$$

Let $L = Df(0) \cdot x$, the linear part of "$\dot{x} = f(x)$, $f(0) = 0$."

Define a linear operator on H_k using L:

$$\begin{aligned} \operatorname{ad} L : \quad H_k &\longrightarrow H_k \\ Y &\longmapsto [Y, L] = DL\,Y - DY\,L \end{aligned}$$

↑ Lie bracket

<u>Example 2.3.2</u> (part I) Let $Df(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$.

Consider $H_2 = \left\{ \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} : P_i = P_i(x,y) = a_i x^2 + b_i xy + c_i y^2,\ \ a_i, b_i, c_i \in \mathbb{R},\ i = 1,2 \right\} \cong \mathbb{R}^6$, a 6-dim'l real vector space spanned by

$$\begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix}.$$

write as $\quad x^2\frac{\partial}{\partial x} \quad xy\frac{\partial}{\partial x} \quad y^2\frac{\partial}{\partial x} \quad x^2\frac{\partial}{\partial y} \quad xy\frac{\partial}{\partial y} \quad y^2\frac{\partial}{\partial y}$

$$\operatorname{ad} L\left(x^2 \frac{\partial}{\partial x}\right) = \operatorname{ad} L \begin{pmatrix} x^2 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{DL}\begin{pmatrix} x^2 \\ 0 \end{pmatrix} - \underbrace{\begin{pmatrix} 2x & 0 \\ 0 & 0 \end{pmatrix}}_{DY}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix} = 2xy\frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y}$$

$$\operatorname{ad} L = 6\times 6\text{-matrix} = \begin{array}{c} \\ x^2\frac{\partial}{\partial x} \\ xy\frac{\partial}{\partial x} \\ y^2\frac{\partial}{\partial x} \\ x^2\frac{\partial}{\partial y} \\ xy\frac{\partial}{\partial y} \\ y^2\frac{\partial}{\partial y} \end{array} \begin{array}{c} \begin{array}{cccccc} x^2\frac{\partial}{\partial x} & xy\frac{\partial}{\partial x} & y^2\frac{\partial}{\partial x} & x^2\frac{\partial}{\partial y} & xy\frac{\partial}{\partial y} & y^2\frac{\partial}{\partial y} \end{array} \\ \left(\begin{array}{ccc|ccc} 0 & -1 & 0 & -1 & 0 & 0 \\ 2 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right) \end{array}$$

Theorem 2.3.3 Let $\dot{x}=f(x)$ be a C^r-system of diff. equations with $f(0)=0$ and $Df(0)x=L$. Let G_k be a complement for the range $\operatorname{ad}L(H_k)$ of $\operatorname{ad}L$ in H_k, i.e. $H_k=\operatorname{ad}L(H_k)+G_k$. Then,

$\exists$ an analytic change of coord. in a nbhd of 0 s.t. $\dot{x}=f(x)$ is transformed to

$$\dot{y}=g(y)=g^{(1)}(y)+g^{(2)}(y)+\dots+g^{(r)}(y)+R_r$$

where $L=g^{(1)}(y)$, $g^{(k)}\in G_k$ for $2\le k\le r$ and $R_r=o(|y|^r)$

Pf: (Sketch), By induction, assume $\dot{x}=f(x)$ is transformed s.t.

$$\dot{x}=f^{(1)}(x)+f^{(2)}(x)+\dots+f^{(s)}(x)+R \quad , \quad L=f^{(1)}(x),\ \text{and } f^{(k)}\in G_k \ \forall\, 2\le k<s$$

Let $x=y+P(y)$ with P being a homog. polyn. of degree s. Then

$$(I+DP(y))\dot{y}=f^{(1)}(y)+f^{(2)}(y)+\dots+f^{(s)}(y)+Df(0)P(y)+o(|y|^s)$$

$$\Big[\ \dot{x}=(I+DP(y))\dot{y}=f^{(1)}(x)+\dots+f^{(s)}(x)+R=f^{(1)}(y+P(y))+\dots+f^{(s)}(y+P(y))+R$$
$$=f^{(1)}(y)+\dots+f^{(s)}(y)+Df(0)P(y)+R'\ \Big]$$

$$\xrightarrow{(I+DP)^{-1}\approx I-DP}\quad \dot{y}=f^{(1)}(y)+\dots+f^{(s)}(y)+\underbrace{Df(0)}_{=DL}P(y)-DP(y)\underbrace{f^{(1)}(y)}_{=L}+o(|y|^s)$$

i.e. the term of degree s is

$$f^{(s)}(y)+DLP(y)-DP(y)L=f^{(s)}(y)+\operatorname{ad}L(P)(y)$$

choose P s.t. the above lies in G_s. #.

Example 2.3.2. (part II) Consider the system

$$\begin{cases} \dot{x} = -y + o(|x|,|y|) \\ \dot{y} = x + o(|x|,|y|) \end{cases} \quad \text{or} \quad \dot{x} = f(x) \quad \text{with} \quad f(0)=0, \; Df(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and we derive the norm form for it:

Let $L = Df(0)\begin{pmatrix} x \\ y \end{pmatrix}$ and $ad L : H_k \to H_k$ as defined above.

$k=2$. as calculated in Example 2.3.2 (part I), $ad L : H_2 \to H_2$ is given by a 6×6-matrix, which is nonsingular, thus $G_2 = \{0\}$. By Thm 2.3.3, all degree-2 terms can be removed from f by change of coord.

$k=3$: Similar calculation shows:

$$ad L = \begin{array}{c|cccccccc} & x^3\frac{\partial}{\partial x} & x^2y\frac{\partial}{\partial x} & xy^2\frac{\partial}{\partial x} & y^3\frac{\partial}{\partial x} & x^3\frac{\partial}{\partial y} & x^2y\frac{\partial}{\partial y} & xy^2\frac{\partial}{\partial y} & y^3\frac{\partial}{\partial y} \\ \hline x^3\frac{\partial}{\partial x} & \multicolumn{8}{c}{\multirow{8}{*}{$\left(\begin{array}{cccc|cccc} 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -3 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right)$}} \\ x^2y\frac{\partial}{\partial x} \\ xy^2\frac{\partial}{\partial x} \\ y^3\frac{\partial}{\partial x} \\ x^3\frac{\partial}{\partial y} \\ x^2y\frac{\partial}{\partial y} \\ xy^2\frac{\partial}{\partial y} \\ y^3\frac{\partial}{\partial y} \end{array}$$

is singular with 2-dimensional kernel. $\Rightarrow G_3 \cong \mathbb{R}^2$, choices of G_3:

Choice 1: $G_3 = \ker(ad L) = \langle v_1, v_2 \rangle = \left\langle \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \\ -3 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\rangle$; v_1, v_2 eigenvectors of 0.

Then, $G_3 = \left\{ a\left(3x^3\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial x} + x^2y\frac{\partial}{\partial y} + 3y^3\frac{\partial}{\partial y}\right) + b\left(x^2y\frac{\partial}{\partial x} + 3y^3\frac{\partial}{\partial x} - 3x^3\frac{\partial}{\partial y} - xy^2\frac{\partial}{\partial y}\right) : a,b\in\mathbb{R} \right\}$

$$= \left\{ \begin{pmatrix} a(3x^3+xy^2)+b(x^2y+3y^3) \\ a(x^2y+3y^3)+b(-3x^3-xy^2) \end{pmatrix} : a,b\in\mathbb{R} \right\}$$

Choice 2 : $G_3 = \left\langle \begin{pmatrix} 1\\0\\1\\0\\0\\1\\0\\1 \end{pmatrix}, \begin{pmatrix} 0\\-1\\0\\-1\\1\\0\\1\\0 \end{pmatrix} \right\rangle$ is also a complement of $\mathrm{ad}L(H_3)$, then

$$G_3 = \left\{ a\left(x^3\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial x} + x^2y\frac{\partial}{\partial y} + y^3\frac{\partial}{\partial y}\right) + b\left(-x^2y\frac{\partial}{\partial x} - y^3\frac{\partial}{\partial x} + x^3\frac{\partial}{\partial y} + xy^2\frac{\partial}{\partial y}\right) : a,b\in\mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} (ax-by)(x^2+y^2) \\ (ay+bx)(x^2+y^2) \end{pmatrix} : a,b\in\mathbb{R} \right\}$$

$\xRightarrow{\text{Thm 2.3.3}}$ the system can be transformed to

$$\begin{cases} \dot{x} = -y + (ax-by)(x^2+y^2) + o(\|(x,y)\|^3) \\ \dot{y} = x + (ay+bx)(x^2+y^2) + o(\|(x,y)\|^3) \end{cases},$$

where the values of a,b are determined by higher derivatives of f.

#

Normal forms for parametrised system:

$$\tilde f(x,\mu) = \begin{pmatrix} f(x,\mu) \\ 0 \end{pmatrix}$$

$\dot x = f(x,\mu)$, $x \in \mathbb{R}^N$, $\mu \in \mathbb{R}$ $\xrightarrow{\text{extend}}$ $\left\{ \begin{array}{l} \dot x = f(x,\mu) \\ \dot\mu = 0 \end{array} \right\}$ $x \in \mathbb{R}^N$, $\mu \in \mathbb{R}$

$f(0,\mu) = 0$ $\forall_\mu$ in a nbhd of $\mu_0 \in \mathbb{R}$

$P = P(y,\mu)$

Consider $\begin{pmatrix} x \\ \mu \end{pmatrix} = H(y,\mu) = \begin{pmatrix} h(y,\mu) \\ \mu \end{pmatrix} = \begin{pmatrix} y + P(y,\mu) \\ \mu + 0 \end{pmatrix}$ for some homog. polyn. of deg. k,

in order to reduce the parametrised system to its "normal form".

Let $L = D_{(x,\mu)} \tilde f(0,\mu_0) \begin{pmatrix} x \\ \mu \end{pmatrix}$ "the linear part of $\left\{ \begin{array}{l} \dot x = f(x,\mu) \\ \dot\mu = 0 \end{array} \right.$"

Let H_k be the lin. space of vec. fields whose coefficients are homogeneous polyn. in $\boxed{(x,\mu)}$ of degree k; i.e.

$$H_k(x,\mu) = \left\{ \begin{pmatrix} P_1 \\ \vdots \\ P_N \\ 0 \end{pmatrix} : P_i \text{ is a homog. polyn. of deg. } k \text{ in } (x_1,\dots,x_N,\mu) \right\}$$

Define $\operatorname{ad} L : H_k(x,\mu) \longrightarrow H_k(x,\mu)$

$$Y = Y(x,\mu) \longmapsto DLY - DYL$$

Theorem 2.3.3' Let $\dot x = f(x,\mu)$ be a C^r-system with $f(0,\mu) = 0$ $\forall_\mu$ near μ_0 and $D_{(x,\mu)} \tilde f(0,\mu_0) \begin{pmatrix} x \\ \mu \end{pmatrix} = L$. If G_k is a complement for $\operatorname{ad} L(H_k)$ in H_k, then $\exists$ analytic change of coord (near 0) s.t. $\dot x = f(x,\mu)$ becomes

$$\dot y = g(y,\mu) = g^{(1)}_{(y,\mu)} + \dots + g^{(r)}_{(y,\mu)} + o(|y|^r) \text{ wh. } L = g^{(1)}(y,\mu),\ g^k \in G_k \ \forall_{2 \le k \le r}.$$

Example 2.3.2 (part II) Consider the param. system

$$\begin{cases} \dot{x} = \mu x - \omega y + o(|x|,|y|) \\ \dot{y} = \omega x + \mu y + o(|x|,|y|) \end{cases} \qquad (x,y)\in\mathbb{R}^2,\ \omega\in\mathbb{R} \text{ fixed constant, } \mu\in\mathbb{R} \text{ parameter}$$

The linearization $\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$ has eigenvalues $\mu \pm i\omega$, so has purely imaginary eigenvalues $\pm i\omega$, for $\mu = \mu_0 = 0$.

Let $L = \left(\begin{array}{cc|c} \cancel{\mu}0 & -\omega & 0 \\ \omega & 0\cancel{\mu} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)\begin{pmatrix} x \\ y \\ \mu \end{pmatrix}$ and $\mathrm{ad}L : H_k \to H_k$. One can show that the system can be brought to the form:

$$(*_p) \begin{cases} \dot{x} = \mu x - \omega y \quad +(ax-by)(x^2+y^2) + O(|(x,y)|^5) \\ \dot{y} = \omega x + \mu y \quad +(ay+bx)(x^2+y^2) + O(|(x,y)|^5) \end{cases},$$

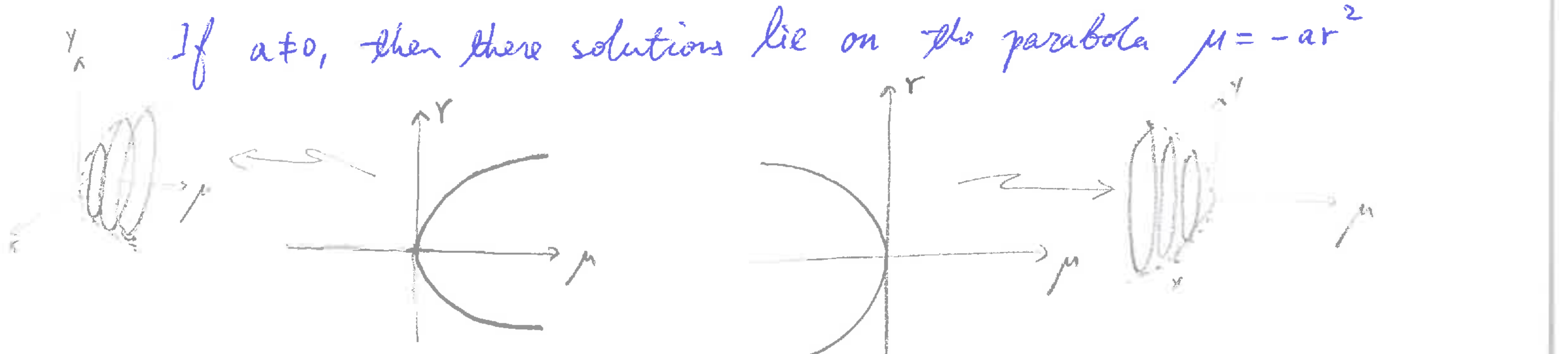
where $a,b \in\mathbb{R}$ are determined by the nonlinearity $o(|(x,y)|^3)$.

Using polar coordinates, we have

$$\begin{cases} \dot{r} = (\mu + ar^2) r \\ \dot{\theta} = \omega + br^2 \end{cases},$$

So periodic solutions can be obtained from $\dot{r} = 0$.

If $a \neq 0$, then these solutions lie on the parabola $\mu = -ar^2$



$a<0$ $\qquad\qquad\qquad\qquad$ $a>0$

Codimension-1 BF :

$$\begin{cases} \dot{x} = f(x,\mu), & x \in \mathbb{R}^N,\ \mu \in \mathbb{R} \\ f(0,0) = 0 \end{cases}$$

1) $D_x f(0,0) \sim \left(\begin{array}{c|c} A & \\ \hline & 0 \end{array} \right)$ $\qquad \sigma(A) \cap i\mathbb{R} = \phi$

↑ I.F.T. for $f(x,\mu)=0$ fails $\rightsquigarrow$ possible steady-state BF.

- **N = 1**

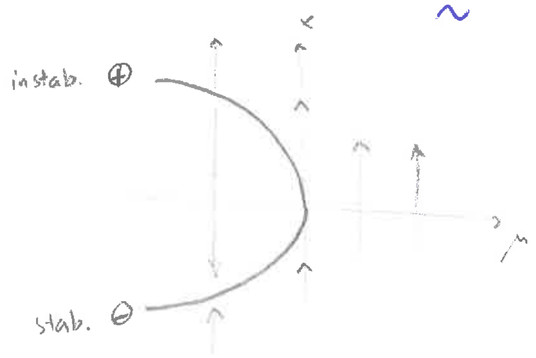
$$f(x,\mu) = \underbrace{f(0,0)}_{=0} + \underbrace{f_x(0,0)}_{=0}\, x + \underbrace{f_\mu(0,0)}_{=a}\,\mu + \underbrace{\tfrac{1}{2} f_{xx}(0,0)}_{=b} \cdot x^2$$
$$+ \tfrac{1}{2} f_{\mu x}(0,0)\,\mu x + \tfrac{1}{2} f_{\mu\mu}(0,0)\,\mu^2 + O(\|(x,\mu)\|^3)$$

(SN1) $a = f_\mu(0,0) \neq 0 \overset{\text{I.F.T.}}{\rightsquigarrow} \mu = \mu(x)$ for $f(x,\mu)=0 \Rightarrow \overset{0}{f_x} + \overset{\neq 0}{f_\mu}\, \mu' = 0 \Rightarrow \begin{rcases} \mu'(0)=0 \\ \mu(0)=0 \end{rcases} \Rightarrow \mu = O(x^2)$

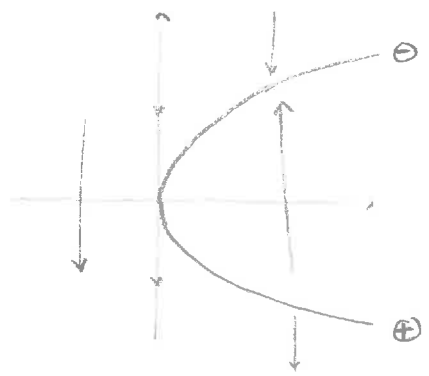
(SN2) $b = \frac{1}{2} f_{xx}(0,0) \neq 0$

$$\Rightarrow \quad f(x,\mu) = a\mu + b x^2 + O(x^3)$$

$$\overset{\text{topo.}}{\sim} \quad \pm\mu \pm x^2$$



$\dot{x} = \mu + x^2$



$\dot{x} = \mu - x^2$

As μ crosses 0, a pair of equilibria with opposite stability annihilate/disappear or come into being/appear, such is called a $\boxed{\text{saddle-node bifurcation}}$.

- <u>general N</u>

Similar conditions:

(SN1) $\langle w, D_\mu f(0,0)\rangle \neq 0$

(SN2) $\langle w, \frac{1}{2} v^T D_{xx} f(0,0) v\rangle \neq 0$

where $v \in \mathbb{R}^N$ is a (right) eigenvector of 0 of $D_x f(0,0)$
and $w \in \mathbb{R}^N$ is a left " " " " "

$T^{-1} A T = J$, $T^{-1} = \begin{pmatrix} w_1^T \\ \vdots \\ w_N^T \end{pmatrix}$, $T = (v_1, \dots, v_N)$, $w_i^T v_j = \delta_{ij}$

$w^T A = \lambda w^T$

<u>Rmk</u>: We say that the equilibrium $(0,0)$ of the system undergoes <u>generically</u> a saddle-node bifurcation under condition that "$D_x f(0,0)$ has a simple critical zero eigenvalue".

Indeed, if a system, say $\dot{x} = f(x,\mu)$, under some equility conditions of derivatives say $f(0,0)=0$ and $\sigma(D_x f(0,0)) \cap i\mathbb{R} = \{\text{simple } 0\}$, will definitely undergo certain phenomenon, in our case S-N BF, if no further equalities of derivatives are satisfied, indicated by (SN1)-(SN2), then the phenomenon is said to be $\boxed{\text{generic}}$, and the non-equality conditions like (SN1)-(SN2) are called $\boxed{\text{generic conditions}}$ for the phenomenon to take place.

Some common cases where (SN1)-(SN2) do not hold :

(1_b) $\quad f(0,\mu)=0 \quad \forall \mu$ near $0 \quad \Rightarrow$ (SN1) fails

or

(1_c) $\quad f(-x,\mu)=-f(x,\mu) \quad \forall x,\mu$ i.e. $\mathbb{Z}_2$-phase symmetry $\Rightarrow$ both (SN1), (SN2) fail

---

(1_b) Transcritical BF.

- $\underline{N=1}$

Let $g(x,\mu)=f_x(x,\mu)$. Then $g(0,0)=0$, $g_x=f_{xx}$, $g_\mu=f_{\mu x}$

$$g(x,\mu)=\underbrace{g(0,0)}_{=0}+\underbrace{g_x(0,0)}_{=2a}x+\underbrace{g_\mu(0,0)}_{=2b}\mu+O(|(x,\mu)|^2)$$

$g_\mu\neq 0 \Rightarrow \mu=\mu(x),\ \mu(0)=0 \Rightarrow \mu=O(x)$

(T1) $a=\frac{1}{2}f_{xx}(0,0)\neq 0$ $\quad\quad \langle w, \frac{1}{2}v^T D_{xx}f(0,0)v\rangle \neq 0$

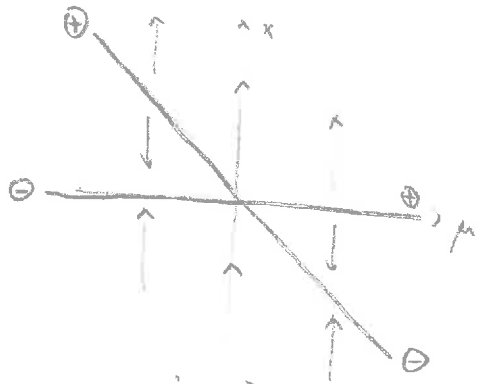
(T2) $b=\frac{1}{2}f_{\mu x}(0,0)\neq 0$ $\quad\quad \langle w, \frac{1}{2}v^T D_{\mu x}f(0,0)v\rangle \neq 0$

for general $N>1$.

$\Rightarrow \quad g(x,\mu)=2ax+2b\mu+O(x^2)$

$\Rightarrow \quad f(x,\mu)=ax^2+2b\mu x+O(x^3)$

$$\overset{\text{topo.}}{\sim} \pm x^2 \pm \mu x$$



$\dot{x}=x^2+\mu x$

A pair of equilibria exchange their stability properties, as μ crosses 0, such is called : $\boxed{\text{transcritical BF.}}$

Rmk: The system $\dot{x} = f(x,\mu)$ with $f(0,\mu) \equiv 0$ generically undergoes a transcritical BF around its equilibrium $(0,0)$ under the simple critical zero eigenvalue condition.

(1c) Pitchfork BF.

- $N=1$.

f odd $\overset{\text{in } x}{\Rightarrow} f_x$ even $\Rightarrow f_{xx}$ odd

Let $g(x,\mu) = f_x(x,\mu)$ Then $g(0,0)=0$, $g_x(0,0) = f_{xx}(0,0) = 0$, $g_\mu = f_{\mu x}$, $g_{xx} = f_{xxx}$

$$g(x,\mu) = \underbrace{g(0,0)}_{0} + \underbrace{g_x(0,0)}_{0} x + \underbrace{g_\mu(0,0)}_{= 2a} \mu + \underbrace{\tfrac{1}{2} g_{xx}(0,0)}_{= 3b} x^2 + \dots$$

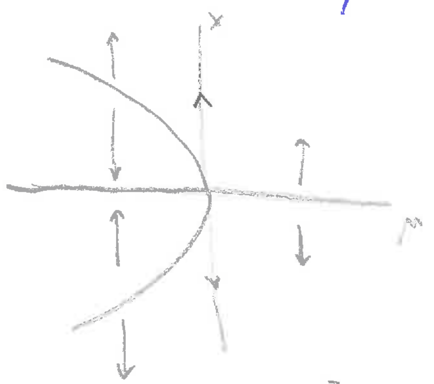
(P1) $a = \frac{1}{2} f_{\mu x}(0,0) \neq 0$: $\langle w, \frac{1}{2} v^T D_{\mu x} f(0,0) v \rangle \neq 0$

(P2) $b = \frac{1}{6} f_{xxx}(0,0) \neq 0$: $\langle w, \frac{1}{6} D_{xxx} f(0,0) (v,v,v) \rangle \neq 0$

$$\Rightarrow \quad g(x,\mu) = 2a\mu + 3b x^2 + \dots$$

$$\Rightarrow \quad f(x,\mu) = 2a\mu x + b x^3 + \dots$$

$$\sim \pm \mu x \pm x^3$$



$\mu x + x^3$

A pair of equilibria of the same stability disappear or appear, while the third equilibrium changes its stability to an opposite one, such is called a $\boxed{\text{pitchfork BF.}}$

Rmk: Pitchfork BF is generic to the equilibrium of the system having $\mathbb{Z}_2$-phase symmetry under the simple critical zero eigenvalue condition.

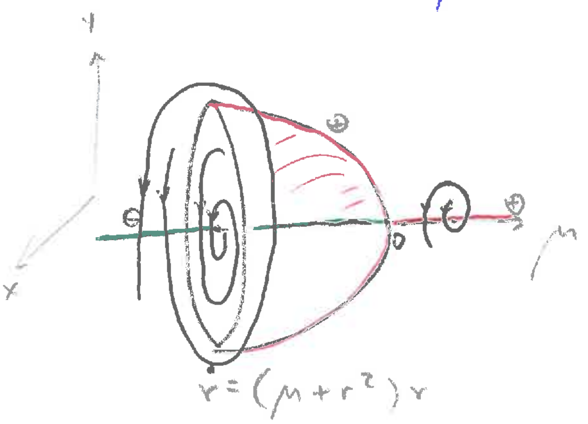
2) $D_x f(0,0) = \left(\begin{array}{cc|c} 0 & -\omega & \\ \omega & 0 & \\ \hline & & A \end{array} \right)$ $\quad \sigma(A) \cap i\mathbb{R} = \phi. \quad \omega \neq 0.$

• N = 2

Using normal form, we have the following normal form :

$$\begin{cases} \dot{x} = (-\omega - b\mu) y + a\mu x + (cx - dy)(x^2+y^2) \\ \dot{y} = a\mu y + (\omega + b\mu) x + (dx + cy)(x^2+y^2) \end{cases} \quad a,b,c,d \in \mathbb{R}.$$

Polar $\Longrightarrow$ $\begin{cases} \dot{r} = (a\mu + cr^2) r \\ \dot{\theta} = \omega + b\mu + dr^2 \end{cases}$ $\quad$ if $\overset{(H1)}{a \neq 0}$, $\overset{(H2)}{c \neq 0}$, then $\dot{r} = (\pm\mu \pm r^2) r$



In general $N \geq 2$; using center mnfd + normal form :

(H1) $a = \frac{d}{d\mu} \operatorname{Re}(\lambda(\mu))\Big|_{\mu=0} \neq 0$

(H2) $c = \frac{1}{16}\left(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right) + \frac{1}{16\omega}\left[f_{xy}(f_{xx}+f_{yy}) - g_{xy}(g_{xx}+g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\right] \neq 0$

<u>Rmk</u>: The equilibrium of a system having one pair of purely imaginary critical eigenvalues undergoes generically a Hopf BF characterized by a birth or disappearance of a surface of per. solutions which lie on the paraboloid $\mu = -\frac{c}{a}(x^2+y^2)$ up to $O(|(x,y)|^3)$.

If $c>0$, they are repelling; if $c<0$, they are attractive.