Center mafd W_p^c and calculations (review)

Consider $\quad \dot{x} = F(x), \quad x \in \mathbb{R}^N$.

Assume it has a fixed point p, w.l.o.G. $p=0$, i.e.

$$F(0)=0.$$

Then, center manifold theorem gives existence of

$$W^s, \quad W^u, \quad W^c$$

stable, unstable, center invariant manifolds around 0, being tangent to

$$E^s, \quad E^u, \quad E^c.$$

Our goal was to calculate W^c and $F|_{W^c}$ (approximately).

Let $J = DF(0)$ and $\sigma(J) = \sigma_s \sqcup \sigma_u \sqcup \sigma_c$. Then, using eigenbasis of J, J can be diagonalized to

$$J = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} : \quad E^u \oplus E^c \oplus E^s \longrightarrow E^u \oplus E^c \oplus E^s$$

$A \quad B \quad C$

For simplicity of calculation, assume $\sigma_u = \phi$.

Thus, $\quad J = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} : \quad E^c \oplus E^s \longrightarrow E^c \oplus E^s$

denote $\dim E^c = n$, $\dim E^s = m$.

$$\dot{x} = F(x), \quad x \in \mathbb{R}^N \xrightarrow[\text{of } J = DF(0)]{\text{eigenbasis}} \begin{cases} \dot{x} = Bx + f(x,y) & (*1) \\ \dot{y} = Cy + g(x,y) & (*2) \end{cases} \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^m, \quad \text{locally around } 0$$

where $f(0,0) = g(0,0) = 0$, $f_x(0,0) = f_y(0,0) = g_x(0,0) = g_y(0,0) = 0$.

$W^c = \{ (x,y) \in E^c \times E^s : y = h(x) \}$ for unknown function $h : U \subset \mathbb{R}^n \to \mathbb{R}^m$.

To determine h:

$$\left.\begin{aligned} (*1) &\Rightarrow \quad \dot{x} = Bx + f(x, h(x)) \\ (*2) &\Rightarrow \quad \dot{y} = Dh(x)\,\dot{x} = Ch(x) + g(x, h(x)) \end{aligned}\right\} \Rightarrow$$

(N): $\quad Dh(x)\big(Bx + f(x, h(x))\big) - Ch(x) - g(x, h(x)) = 0 \quad$ with $h(0) = 0$, $Dh(0) = 0$

assuming h is a polynomial of x, we determine h by comparing coefficients of x^n in (N).

Example 2.2.2 (part I) Consider the Lorenz system:

$$(L) \quad \begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = -\beta z + xy \end{cases} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3, \quad \sigma, \rho, \beta > 0 \text{ parameters.}$$

Consider when $\rho = 1$ and BF at $(0,0,0)$:

$$J = DF(0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix} \qquad \sigma(J) = \{ 0, \underset{<0}{-\sigma-1}, \underset{<0}{-\beta} \}$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} \sigma \\ -1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{1+\sigma} & \frac{\sigma}{1+\sigma} & 0 \\ \frac{1}{1+\sigma} & \frac{-1}{1+\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{P} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\qquad P^{-1} J P = \Lambda$

$$\begin{pmatrix} \dot u \\ \dot v \\ \dot w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sigma-1 & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} \overbrace{-\frac{\sigma}{1+\sigma}(u+\sigma v)w}^{f} \\ \underbrace{\begin{matrix} \frac{1}{1+\sigma}(u+\sigma v)w \\ (u+\sigma v)(u-v) \end{matrix}}_{g} \end{pmatrix} \qquad (*)$$

E^c = u-axis, W^c is tangent to E^c at 0.

$W^c = \{ (u,v,w) \in E^c \times E^s : v = h_1(u),\ w = h_2(u) \}$ $\qquad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} : U \subset \mathbb{R} \to \mathbb{R}^2$

(N) becomes: $B = 0$, $C = \begin{pmatrix} -\sigma-1 & 0 \\ 0 & -\beta \end{pmatrix}$

$$\begin{pmatrix} h_1'(u) \\ h_2'(u) \end{pmatrix} \left(-\frac{\sigma}{1+\sigma}(u+\sigma h_1(u)) h_2(u) \right) - \begin{pmatrix} -\sigma-1 & 0 \\ 0 & -\beta \end{pmatrix} \begin{pmatrix} h_1(u) \\ h_2(u) \end{pmatrix} - \begin{pmatrix} \frac{1}{1+\sigma}(u+\sigma h_1(u)) h_2(u) \\ (u+\sigma h_1(u))(u - h_1(u)) \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} -\frac{\sigma}{\cancel{1+\sigma}}(u+\sigma h_1(u)) h_2(u) h_1'(u) + (\sigma+1)^2 h_1(u) - \frac{1}{\cancel{1+\sigma}}(u+\sigma h_1(u)) h_2(u) = 0 \\ -\frac{\sigma}{\cancel{1+\sigma}}(u+\sigma h_1(u)) h_2(u) h_2'(u) + {}^{(1+\sigma)}\beta h_2(u) - {}^{(1+\sigma)}(u+\sigma h_1(u))(u - h_1(u)) = 0 \end{cases}$$

Let $h_1(u) = a_1 u^2 + b_1 u^3 + c_1 u^4 + \dots$

$\ h_2(u) = a_2 u^2 + b_2 u^3 + c_2 u^4 + \dots$

$$-\sigma(u + \sigma a_1 u^2 + \sigma b_1 u^3 + \sigma c_1 u^4 + \dots)(a_2 u^2 + b_2 u^3 + c_2 u^4 + \dots)(2 a_1 u + 3 b_1 u^2 + 4 c_1 u^3 + \dots)$$

$$+ (\sigma+1)^2 (a_1 u^2 + b_1 u^3 + c_1 u^4 + \dots) - (u + \sigma a_1 u^2 + \sigma b_1 u^3 + \sigma c_1 u^4 + \dots)(a_2 u^2 + b_2 u^3 + c_2 u^4 + \dots) = 0$$

u^2-coeff: $(\sigma+1)^2 a_1 = 0 \quad \Rightarrow a_1 = 0$

u^3-coeff: $(\sigma+1)^2 b_1 - a_2 = 0 \quad \Rightarrow b_1 = \frac{a_2}{(\sigma+1)^2} = \frac{1}{\beta(\sigma+1)^2}$

$$-\sigma\left(u+\sigma a_1 u^2+\sigma b_1 u^3+\sigma c_1 u^4+\dots\right)\left(a_2 u^2+b_2 u^3+c_2 u^4+\dots\right)\left(2a_2 u+3b_2 u^2+4c_2 u^3+\dots\right)$$

$$+(1+\sigma)\beta\left(a_2u^2+b_2u^3+c_2u^4+\dots\right)-(1+\sigma)\left(u+\sigma a_1u^2+\sigma b_1u^3+\sigma c_1u^4+\dots\right)\left(u-a_1u^2-b_1u^3-c_1u^4-\dots\right) = 0.$$

u^2-coeff: $\quad (1+\sigma)\beta a_2-(1+\sigma)=0 \Rightarrow a_2=\frac{1}{\beta}$

u^3-coeff: $\quad (1+\sigma)\beta b_2-(1+\sigma)(-a_1+\sigma a_1)=0 \overset{a_1=0}{\Longrightarrow} b_2=0.$

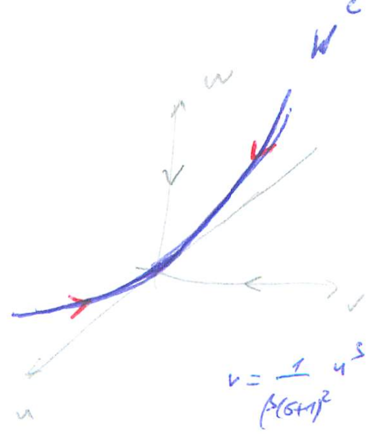
$$\Rightarrow \quad h_1(u)=\frac{1}{\beta(\sigma+1)^2}u^3+O(u^4), \qquad h_2(u)=\frac{u^2}{\beta}+O(u^4)$$

On W^c:

$$\dot u = -\frac{\sigma}{1+\sigma}\left(u+\sigma h_1(u)\right)h_2(u) \doteq -\frac{\sigma}{1+\sigma}\left(u+\frac{\sigma}{\beta(\sigma+1)^2}u^3\right)\cdot\frac{u^2}{\beta}$$

$$= \underbrace{\boxed{-\frac{\sigma}{(1+\sigma)\beta}}}_{<0}u^3-\frac{\sigma^2}{(1+\sigma)^3\beta^2}u^5$$

weakly attracting



$v=\frac{1}{\beta(\sigma+1)^2}u^3$

$w=\frac{u^2}{\beta}$

(to be continued).

Calculation of W^c for parametrised systems:

Consider $\quad \dot{x} = F_\mu(x) \quad , \quad x \in \mathbb{R}^N, \ \mu \in \mathbb{R}^k$ parameter.

Assume $\quad F_\mu(0) = 0 \quad \forall_\mu$ in a nbhd of $\mu_0 \in \mathbb{R}^k$.

Set $\quad J := J_{\mu_0} = DF_{\mu_0}(0)$. Assume $\sigma(J) \cap i\mathbb{R} \neq \phi$.

We derive W_0^c for $\mu = \mu_0$ as follows:

- using eigenbasis write $J = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, where $Re(\lambda) < 0 \ \forall_{\lambda \in \sigma(C)}$, $Re(\lambda) = 0 \ \forall_{\lambda \in \sigma(B)}$

  (we assume here $\sigma(J) = \sigma_s \sqcup \sigma_c \sqcup \sigma_u$ with $\sigma_u = \phi$), and the system can be rewritten as (w.r.t. eigenbasis)

$$\begin{cases} \dot{x} = B_\mu x + f_\mu(x,y) \\ \dot{y} = C_\mu y + g_\mu(x,y) \end{cases}, \quad \text{where } B_{\mu=\mu_0} = B, \ C_{\mu=\mu_0} = C.$$

- consider

$$\begin{cases} \dot{x} = B_\mu x + f_\mu(x,y) \\ \dot{\mu} = 0 \\ \dot{y} = C_\mu y + g_\mu(x,y) \end{cases} \iff \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{\mu} \end{pmatrix} = \overbrace{\boxed{\begin{pmatrix} B_\mu & 0 \\ 0 & 0 \end{pmatrix}}}^{B} \begin{pmatrix} x \\ \mu \end{pmatrix} + \underbrace{\begin{pmatrix} f_\mu(x,y) \\ 0 \end{pmatrix}}_{f} \\ \dot{y} = \underbrace{\boxed{C_\mu}}_{C} y + \underbrace{g_\mu(x,y)}_{g} \end{cases}$$

One can apply "(N): $Dh(x)(Bx + f(x,h(x))) - C \cdot h(x) - g(x,h(x)) = 0$"

where $x = \begin{pmatrix} x \\ \mu \end{pmatrix}$ and $h = h(x,\mu) = ax^2 + b\mu^2 + cx\mu + O(\|x,\mu\|^3)$, to determine

$$W^c = \{(x,y,\mu) \in \mathbb{R}^N \times \mathbb{R}^k : \ y = h(x,\mu)\}.$$

Example 2.2.2. (continued.) (part II) Consider the Lorenz system near $\rho = 1$:

$$(L) \begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = -\beta z + xy \end{cases} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3, \quad \sigma, \beta > 0, \quad \rho \text{ near } 1.$$

Using the same eigenbasis-transformation:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\sigma} & \frac{\sigma}{1+\sigma} & 0 \\ \frac{1}{1+\sigma} & \frac{-1}{1+\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{i.e.} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

we have (L):

$$\begin{cases} \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \frac{\sigma(\rho-1)}{1+\sigma} & \frac{\sigma^2(\rho-1)}{1+\sigma} & 0 \\ -\frac{\rho-1}{1+\sigma} & -\frac{\sigma(\sigma+\rho)}{1+\sigma} - 1 & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} -\frac{\sigma}{1+\sigma}(u+\sigma v)w \\ \frac{1}{1+\sigma}(u+\sigma v)w \\ (u+\sigma v)(u-v) \end{pmatrix} \\ \text{together with} \\ \dot{\rho} = 0 \end{cases}$$

At $(u, v, w, \rho) = (0, 0, 0, 1)$, the above system has center manifold

$$E^c = \text{"}u\text{-axis"} \times \text{"}\rho\text{-axis"} \simeq \mathbb{R}^2 \subseteq \mathbb{R}^4$$

thus $W^c = \{ (u, v, w, \rho) : v = h_1(u, \rho), w = h_2(u, \rho) \}$ $\quad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2.$

Rewrite the system as:

$$
\begin{cases}
\begin{pmatrix} \dot u \\ \dot \rho \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\sigma(\rho-1)}{1+\sigma} & 0 \\ 0 & 0 \end{pmatrix}}_{B} \begin{pmatrix} u \\ \rho \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{\sigma^2(\rho-1)}{1+\sigma} v - \frac{\sigma}{1+\sigma}(u+\sigma v) w \\ 0 \end{pmatrix}}_{f} \\[2ex]
\begin{pmatrix} \dot v \\ \dot w \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{\sigma(\sigma+\rho)}{1+\sigma} - 1 & 0 \\ 0 & -\beta \end{pmatrix}}_{C} \begin{pmatrix} v \\ w \end{pmatrix} + \underbrace{\begin{pmatrix} -\frac{\rho-1}{1+\sigma} u + \frac{1}{1+\sigma}(u+\sigma v) w \\ (u+\sigma v)(u-v) \end{pmatrix}}_{g}
\end{cases}
$$

(N) becomes:

$$
\begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial \rho} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial \rho} \end{pmatrix}
\left[\begin{pmatrix} \frac{\sigma(\rho-1)}{1+\sigma} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ \rho \end{pmatrix} + \begin{pmatrix} \frac{\sigma^2(\rho-1)}{1+\sigma} h_1 - \frac{\sigma}{1+\sigma}(u+\sigma h_1) h_2 \\ 0 \end{pmatrix} \right] - \begin{pmatrix} -\frac{\sigma(\sigma+\rho)}{1+\sigma} - 1 & 0 \\ 0 & -\beta \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}
$$

$$
- \begin{pmatrix} -\frac{\rho-1}{1+\sigma} u + \frac{1}{1+\sigma}(u+\sigma h_1) h_2 \\ (u+\sigma h_1)(u-h_1) \end{pmatrix} = 0.
$$

$$
\Rightarrow
\begin{cases}
\frac{\partial h_1}{\partial u} \cdot \left[\frac{\sigma(\rho-1)}{1+\sigma} u + \underbrace{\frac{\sigma^2(\rho-1)}{1+\sigma} h_1 - \frac{\sigma}{1+\sigma}(u+\sigma h_1) h_2}_{} \right] + \left(\frac{\sigma(\sigma+\rho)}{1+\sigma} + 1 \right) h_1 + \frac{\rho-1}{1+\sigma} u - \frac{1}{1+\sigma}(u+\sigma h_1) h_2 = 0 \\[2ex]
\frac{\partial h_2}{\partial u} \left[\frac{\sigma(\rho-1)}{1+\sigma} u + {''} \right] + \beta h_2 - (u+\sigma h_1)(u-h_1) = 0
\end{cases}
$$

Let $h_1(u,\rho) = a_1 u^2 + b_1 u \tilde\rho + c_1 \tilde\rho^2 + d_1 u^3 + \dots$

where $\tilde\rho = \rho - 1$

$h_2(u,\rho) = a_2 u^2 + b_2 u \tilde\rho + c_2 \tilde\rho^2 + d_2 u^3 + \dots$

$$
(2a_1 u + b_1 \tilde\rho + 3 d_1 u^2 + \dots) \Big[\frac{\sigma(\rho-1)}{1+\sigma} u + \frac{\sigma^2(\rho-1)}{1+\sigma} (a_1 u^2 + b_1 u \tilde\rho + c_1 \tilde\rho^2 + d_1 u^3 + \dots) - \frac{\sigma}{1+\sigma} (u + \sigma a_1 u^2 + \sigma b_1 u \tilde\rho + \sigma c_1 \tilde\rho^2
$$

$$
+ \sigma d_1 u^3 + \dots)(a_2 u^2 + b_2 u \tilde\rho + c_2 \tilde\rho^2 + d_2 u^3 + \dots) \Big] + \left(\frac{\sigma(\sigma+\rho)}{1+\sigma} + 1 \right) (a_1 u^2 + b_1 u \tilde\rho + c_1 \tilde\rho^2 + d_1 u^3 + \dots) + \frac{\rho-1}{1+\sigma} u
$$

$$
- \frac{1}{1+\sigma} (u + \sigma a_1 u^2 + \sigma b_1 u \tilde\rho + \sigma c_1 \tilde\rho^2 + \sigma d_1 u^3 + \dots)(a_2 u^2 + b_2 u \tilde\rho + c_2 \tilde\rho^2 + d_2 u^3 + \dots) = 0
$$

$\underline{u^2}$: $\left(\frac{\sigma(\sigma+\rho)}{1+\sigma}+1\right) a_1 = 0 \Rightarrow a_1 = 0.$

$\underline{\underline{u\tilde\rho}}$: $\left(\frac{\sigma(\sigma+\rho)}{1+\sigma}+1\right) b_1 = 0 \Rightarrow b_1 = 0.$

$\tilde\rho^2$: $\left(\frac{\sigma(\sigma+\rho)}{1+\sigma}+1\right) c_1 = 0 \Rightarrow c_1 = 0.$

u^3: $\left(\frac{\sigma(\sigma+\rho)}{1+\sigma}+1\right) d_1 - \frac{1}{1+\sigma}\cdot a_2 = 0 \Rightarrow d_1 = \frac{a_2}{\sigma(\sigma+\rho)+\sigma+1} = \frac{1}{\beta(\sigma(\sigma+\rho)+\sigma+1)}$

$$\left(2a_2 u + b_2\tilde\rho + 3d_2u^2 + \dots\right)\left[\frac{\sigma(\rho-1)}{1+\sigma}u + \frac{\sigma^2(\rho-1)}{1+\sigma}\left(a_1u^2 + b_1u\tilde\rho + c_1\tilde\rho^2 + d_1u^3 + \dots\right) - \frac{\sigma}{1+\sigma}\left(u + \sigma a_1u^2 + \sigma b_1u\tilde\rho + \sigma c_1\tilde\rho^2 + \dots\right)\right.$$

$$\left.\cdot\left(a_2u^2 + b_2u\tilde\rho + c_2\tilde\rho^2 + d_2u^3 + \dots\right)\right] + \beta a_2u^2 + \beta b_2u\tilde\rho + \beta c_2\tilde\rho^2 + \beta d_2u^3 + \dots - \left(u + \sigma a_1u^2 + \sigma b_1u\tilde\rho + \sigma c_1\tilde\rho^2 + \dots\right)$$

$$\cdot\left(u - a_1u^2 - b_1u\tilde\rho - c_1\tilde\rho^2 - d_1u^3 - \dots\right) = 0.$$

$\underline{u^2}$: $\beta a_2 - 1 = 0 \Rightarrow a_2 = \frac{1}{\beta}$

$\underline{\underline{u\tilde\rho}}$: $\beta b_2 = 0 \Rightarrow b_2 = 0$

$\tilde\rho^2$: $\beta c_2 = 0 \Rightarrow c_2 = 0.$

$$h_1(u,\rho) = \frac{1}{\beta(\sigma(\sigma+\rho)+\sigma+1)}u^3 + O(u^4)$$

$$h_2(u,\rho) = \frac{1}{\beta}u^2 + O(u^3)$$

$$\Rightarrow \quad \dot u = \frac{\sigma(\rho-1)}{1+\sigma}u + \frac{\sigma^2(\rho-1)}{1+\sigma}h_1 - \frac{\sigma}{1+\sigma}(u+\sigma h_1)h_2$$

$$= \frac{\sigma(\rho-1)}{1+\sigma}u + \frac{\sigma^2(\rho-1)}{1+\sigma}\,\frac{1}{\beta(\sigma(\sigma+\rho)+\sigma+1)}u^3 - \underbrace{\frac{\sigma}{1+\sigma}\left(u + \frac{\sigma}{\beta(\dots)}u^3\right)\cdot\frac{1}{\beta}u^2}_{-\frac{\sigma}{\beta(1+\sigma)}u^3 - \dots}$$

$$= \frac{\sigma(\rho-1)}{1+\sigma}u + \frac{\sigma^2(\rho-1) - \sigma(\dots)}{(1+\sigma)\beta(\dots)}u^3 + \dots$$

$$\underbrace{\frac{\sigma^2(\rho-1) - \sigma(\dots)}{(1+\sigma)\beta(\dots)}}_{} = \frac{\sigma^2(\rho-1) - \sigma(\sigma(\sigma+\rho)+\sigma+1)}{(1+\sigma)\beta(\dots)} = \frac{\sigma^2\rho - \sigma^2 - \sigma^3 - \sigma^2\rho - \sigma^2 - \sigma}{''} = \frac{-\sigma^3 - 2\sigma^2 - \sigma}{''}$$

$$= \frac{-\sigma(\sigma+1)^{\cancel{2}}}{\cancel{(1+\sigma)}\beta(\dots)} = \frac{-\sigma(\sigma+1)}{\beta(\sigma(\sigma+\rho)+\sigma+1)} < 0.$$

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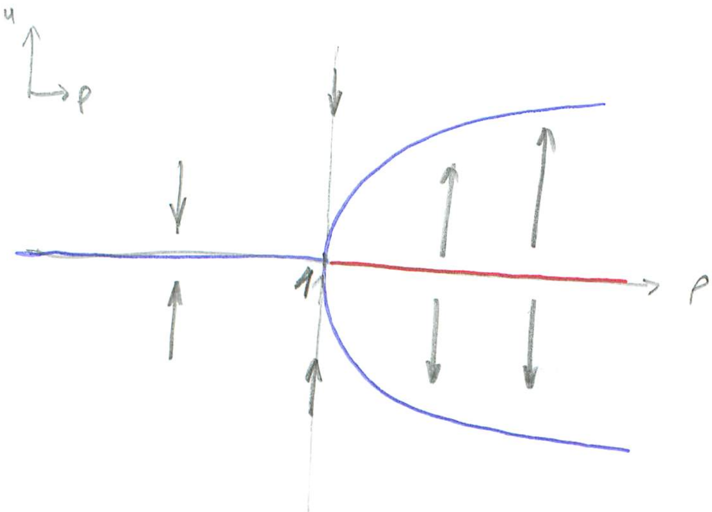
Essentially, on W^c, we have

$$\dot{u} = (\rho - 1)u - u^3 \qquad (\text{compare with Example 2.1.3})$$



when $\rho < 1$, $u = 0$ is the only FP, and it is stable,

when $\rho > 1$, $u = \pm\sqrt{\rho - 1}$ are additional FPs, and they are stable, $u = 0$ becomes unstable.

when $\rho = 1$, $\dot{u} = -u^3$, $\Rightarrow$ $u = 0$ is the only FP, and is weakly stable (this coincides with our calculation in Example 2.2.2, (part I)).

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