A relatively complete bifurcation theory is available for:

Bifurcation of Equilibria:

Consider the set of all equilibria of $(*)$:

$$E = \{ (\mu, x) : \quad f(\mu, x) = 0 \} \quad , \quad \text{let } (\mu_0, x_0) \in E.$$

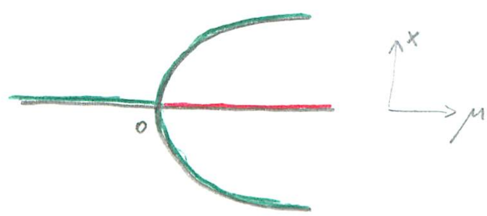
For those parameter values μ_0, at which $D_x f(\mu_0, x_0)$ is invertible
i.e. $0 \notin \sigma(D_x f(\mu_0, x_0))$,

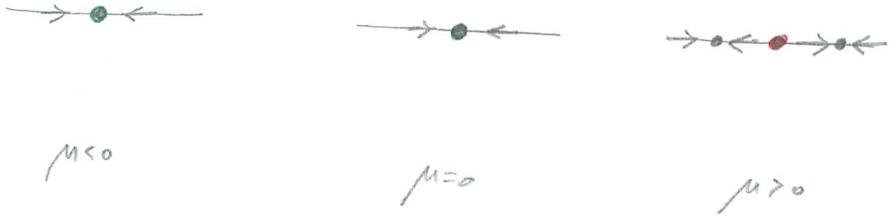
the equilibria can be smoothly parametrized by μ: (I.F.T.)

$$E \cap B_\varepsilon(\mu_0, x_0) = \{ (\mu, x(\mu)) : \mu \in B_\varepsilon(\mu_0) \subset \mathbb{R}^k, \ x(\mu_0) = x_0 \}$$ one "branch" of equilibria

Otherwise, when $D_x f(\mu_0, x_0)$ is not invertible, around (μ_0, x_0) several
necessary condition for BF, not sufficient (HW)
branches of equilibria can come together at (μ_0, x_0).

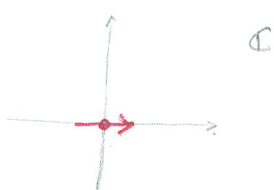
Example 2.1.3 $\quad f(\mu, x) = \mu x - x^3, \quad x \in \mathbb{R}, \mu \in \mathbb{R}$





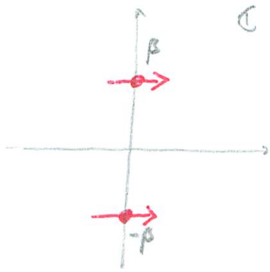
"pitchfork BF"

From the point of stability, one can say that this bifurcation arises as a result of change of stability of equilibrium $x = 0$.



corresponding change of spectrum of $D_x f(\mu, x)$.

Another way an equilibrium can change its stability is



$D_x f(\mu_0, x_0)$ is invertible but $\pm i\beta \in \sigma(D_x f(\mu_0, x_0))$.

By I.F.T., around (μ_0, x_0), $(*)$ has a unique branch of equilibria $(\mu, x(\mu))$, but other types of sol.s may arise such as periodic solutions.

Example 2.1.4 $\begin{cases} \dot{x} = \mu x - y - x(x^2+y^2) \\ \dot{y} = x + \mu y - y(x^2+y^2) \end{cases}$ written as $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(\mu, \begin{pmatrix} x \\ y \end{pmatrix})$, $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

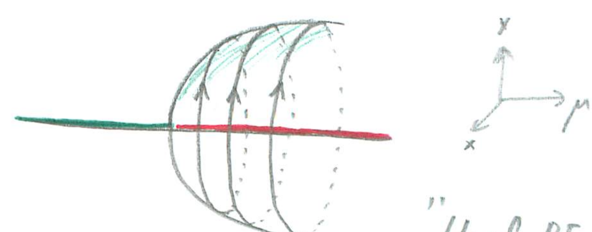
Consider the stability of the equilibrium $(0,0) \in \mathbb{R}^2$, when μ varies:

$$D_{(x,y)} f(\mu, (0,0)) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} \text{ has eigenvalues } \mu \pm i;$$

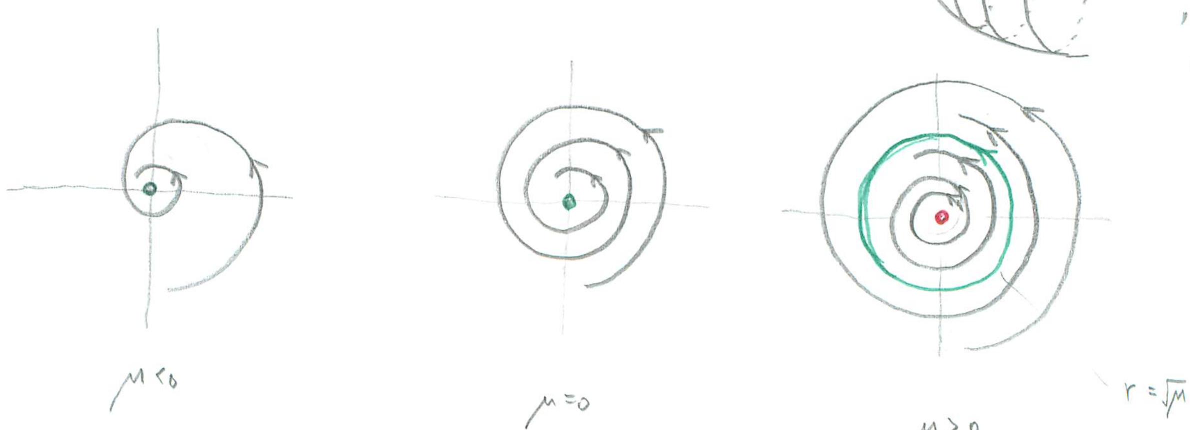
So as μ crosses 0, one pair of eigenvalues of $D_{(x,y)} f(\mu, (0,0))$ cross $\pm i$.

Using polar coordinates, we have

$$\begin{cases} \dot{r} = (\mu - r^2) r \\ \dot{\theta} = 1 \end{cases}$$



"Hopf-BF"



New branches of per. solutions arise as a result of change of stability of the equilibrium $x=0$. (Hopf BF)

In the context of bifurcation of equilibria due to change of stability of the equilibria, we call 0 (resp. $\pm i\beta$) critical eigenvalues of the linearization $D_x f(\mu_0, x_0)$ corresponding to the bifurcation value μ_0.

Codimension of Bifurcation of Equilibria

- Given a bifurcation value μ_0, by I.F.T, $D_x f(\mu_0, x_0)$ has critical eigenvalues: 0 and/or $\pm i\beta$. Depending on the type of the critical eigenvalues, we define the codimension of the bifurcation:

Example 2.1.5 Let $M_{2\times 2}$ be the space of 2×2 matrices. Consider the set

$$S = \{ A \in M_{2\times 2} : A \text{ has } 0 \text{ as a simple eigenvalue} \}$$

- Then, $S \subset M_{2\times 2}$ is a submanifold of codimension 1 in $M_{2\times 2} \cong \mathbb{R}^4$

Pf: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\det(\lambda I_{2\times 2} - A) = \det\begin{pmatrix} \lambda-a & -b \\ -c & \lambda-d \end{pmatrix} = \lambda^2 - (a+d)\lambda + ad-bc = 0 \quad \Rightarrow \begin{cases} \lambda_1+\lambda_2 = a+d \\ \lambda_1 \cdot \lambda_2 = ad-bc \end{cases}$$

$\lambda = 0$ is an eigenvalue $\Rightarrow$ $\lambda_1 \cdot \lambda_2 = 0 \Rightarrow ad-bc = 0$

$\lambda = 0$ is simple $\Rightarrow$ $0 = \lambda_1 \neq \lambda_2$ $\Rightarrow a+d \neq 0$

Define $F: \quad M_{2\times 2} \longrightarrow \mathbb{R}$ then $S \subseteq F^{-1}(0)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto ad-bc$$

(HW) for other types of eigenvalues

$DF(a,b,c,d) = (d, -c, -b, a)$; if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$, then $a+d \neq 0$, so $\operatorname{rank} DF(\ldots) = 1$

$\Rightarrow$ every point of S is a regular point of F. $\Rightarrow \dim S = 3 \Rightarrow \operatorname{codim} S = 1$.

S is a submfd of

Def. 2.1.6 Suppose that (*) undergoes a bifurcation of equilibria at μ_0 and $D_x f(\mu_0, x_0)$ has critical eigenvalues of type (T) (such as one simple eigenvalue 0, + one simple pair of $\pm i$). Then, we say the bifurcation is of codimension m, if

$$\{ A \in M_{n\times n} : A \text{ has eigenvalues of type (T)} \} \subseteq M_{n \times n}$$

is a submanifold of codimension m in $M_{n\times n}$.

Some examples are: (HW)

Codimension one

(i) simple zero eigenvalue: $D_x f_\mu = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$

(ii) simple pure imaginary pair: $D_x f_\mu = \begin{pmatrix} \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} & 0 \\ 0 & A \end{pmatrix}$

Codimension two

(iii) double zero, nondiagonalizable: $D_x f_\mu = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & A \end{pmatrix}$

(iv) simple zero, simple pair of pure. imag. $D_x f_\mu = \begin{pmatrix} \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & 0 \\ 0 & A \end{pmatrix}$

(v) two distinct pairs of pur. imag. $D_x f_\mu = \begin{pmatrix} \left(\begin{array}{cc|cc} 0 & -\omega_1 & \multicolumn{2}{c}{0} \\ \omega_1 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & -\omega_2 \\ & & \omega_2 & 0 \end{array}\right) & 0 \\ 0 & A \end{pmatrix}$,

where A has no eigenvalues on $i\mathbb{R}$.

2.2. Center Manifolds

Theorem 2.2.1. Let f be a C^r-vector field on $\mathbb{R}^n$ vanishing at the origin ($f(0)=0$) and let $A = Df(0)$. Divide the spectrum of A into three parts $\sigma_s, \sigma_c, \sigma_u$ with

$$\mathrm{Re}\,\lambda \begin{cases} <0 & \text{if } \lambda\in\sigma_s \\ =0 & \text{if } \lambda\in\sigma_c \\ >0 & \text{if } \lambda\in\sigma_u . \end{cases}$$

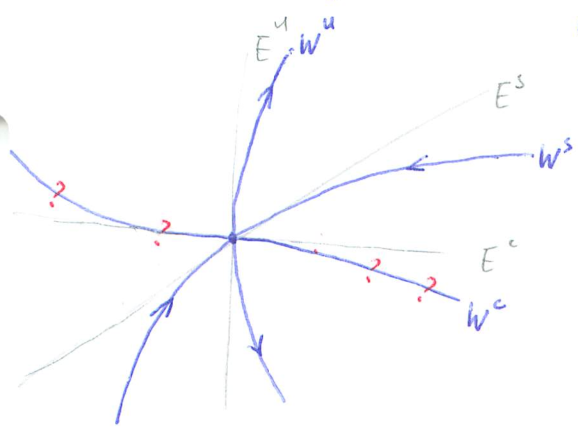
Let the (generalized) eigenspaces of σ_s, σ_c and σ_u be E^s, E^c and E^u, respectively.

Then,

(i) $\exists$ C^r stable and unstable inv. mfds W^s and W^u tangent to E^s and E^u at 0, respectively;

(ii) $\exists$ C^{r-1} center manifold W^c tangent to E^c at 0.

(iii) W^s and W^u are unique, but W^c need not be. (HW)



Calculation of center mnfd W_ρ^c:

By Thm 2.2.1 (center mnfd theorem), bifurcation of system

(*) $\dot{x} = F(x)$, $F(0)=0$, $J = DF(0)$

at 0 is topologically equivalent to

$$\begin{cases} \dot{\tilde{x}} = \tilde{f}(x) \\ \dot{\tilde{y}} = -\tilde{y} \\ \dot{\tilde{z}} = \tilde{z} \end{cases} \qquad (\tilde{x}, \tilde{y}, \tilde{z}) \in W^c \times W^s \times W^u$$

For simplicity, assume $W^u = \phi$. We want to compute $\tilde{f}$ (approximately)!

Let $n = \dim W^c$, $m = \dim W^s$. Then, the linearization

$$J = DF(0) = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, \quad \text{with respect to eigenbasis in } E^c \times E^s$$

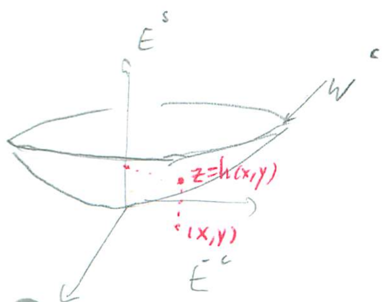
for some $B \in M_{n\times n}$ whose eigenvalues all have zero real parts, and some $C \in M_{m\times m}$ whose eigenvalues all have negative real parts.

And (*) becomes w.r.t. $\mathbb{R}^{n+m} = E^c \times E^s \ni (x, y)$

$$(D) \begin{cases} \dot{x} = Bx + f(x,y) & (D1) \\ \dot{y} = Cy + g(x,y) & (D2) \end{cases} \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^m,$$

where $f(0,0) = g(0,0) = 0$, $f_x(0,0) = f_y(0,0) = g_x(0,0) = g_y(0,0) = 0$ since $F(0) = 0$, $J = DF(0) = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$

Thus, the center mnfd W^c at 0 is tangent to $E^c = \{(x,y) : y = 0\}$,



So W^c can be parametrized using $x \in E^c$:

$W^c = \{(x,y) \in E^c \times E^u : y = h(x)\}$ for some $h : U \subset \mathbb{R}^n \to \mathbb{R}^m$ with

$h(0) = 0$, $Dh(0) = 0$.
$(0,0) \in W^c$ $\quad$ W^c tangent to $\{(x,0)\}$

$$(D1) \overset{y = h(x)}{\Longrightarrow} \dot{x} = Bx + f(x, h(x))$$

If we can compute h, then dynamics of (D1) around 0 is a good approximate of dynamics of (D) around 0.

<u>Theorem 2.2.3</u> If $x = 0$ of (D1) is locally asymptotically stable (resp. unstable), then $\binom{x}{y} = \binom{0}{0} = 0$ of (D) is also loc. asympt. stable (resp. unstable).

To compute h:

$$(D2) \overset{y=h(x)}{\Longrightarrow} \quad \dot{y} = Dh(x)\,\dot{x} = Dh(x)\left(Bx + f(x,h(x))\right) = Ch(x) + g(x,h(x))$$

$$\Rightarrow \quad \underbrace{Dh(x)\left(Bx+f(x,h(x))\right) - Ch(x) - g(x,h(x))}_{\mathcal{N}(h(x))} = 0 \quad (N)$$

with bd. cond. $h(0)=0,\ Dh(0)=0$.

<u>Theorem 2.2.4</u> If there exists $\phi(x)$ with $\phi(0) = D\phi(0) = 0$ s.t.

$\mathcal{N}(\phi(x)) = O(|x|^p)$ for some $p>1$ as $|x|\to 0$, then $h(x) = \phi(x) + O(|x|^p)$ as $|x|\to 0$.

Thus, the solution of (N) <u>can</u> be approximated arbitrarily closely as a Taylor series at $x=0$.

<u>Example 2.2.5</u> Consider
$$\begin{cases} \dot{u} = v \\ \dot{v} = -v + \alpha u^2 + \beta uv \end{cases}$$
$(u,v)\in\mathbb{R}^2$, $\alpha,\beta\in\mathbb{R}$ parameters.

$J = Df(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$ has eigenvalues 0 and -1.
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Let $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, we transform the system to:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ \alpha(x+y)^2 - \beta(x+y)y \end{pmatrix}$$

i.e.
$$\begin{cases} \dot{x} = \underbrace{\alpha(x+y)^2 - \beta(xy+y^2)}_{=f(x,y)} \\ \dot{y} = -y \underbrace{- \alpha(x+y)^2 + \beta(xy+y^2)}_{=g(x,y)} \end{cases}$$

then (N) becomes:
$B=0,\ C=-1$

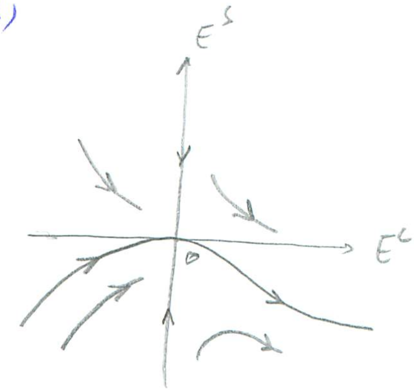
$$\begin{cases} h'(x)\left(\alpha(x+h(x))^2 - \beta(xh(x)+h(x)^2)\right) + h(x) + \alpha(x+h(x))^2 - \beta(xh(x)+h(x)^2)) = 0 \\ h(0) = h'(0) = 0. \end{cases}$$

Let $h(x) = ax^2 + bx^3 + \dots$ ($\Rightarrow h'(x) = 2ax + 3bx^2 + \dots$), then

$$(2ax+3bx^2+\dots)\big(\alpha(x+ax^2+bx^3+\dots)^2 - \beta(ax^3+bx^4+a^2x^4+2abx^5+\dots)\big)$$
$$+\, ax^2+bx^3+\dots+\alpha(x+ax^2+bx^3+\dots)^2 - \beta(ax^3+bx^4+a^2x^4+2abx^5+\dots) = 0$$

x^2-coeff: $a+\alpha=0$ $a=-\alpha$

x^3-coeff: $2a\alpha + b + 2a\alpha - a\beta = 0 \Rightarrow b = 4\alpha^2 - \alpha\beta = \alpha(4\alpha-\beta)$



$$\Rightarrow h(x) = -\alpha x^2 + \alpha(4\alpha-\beta)x^3 + O(x^4)$$

$$\Rightarrow \dot x = \alpha(x+h(x))^2 - \beta(xh(x)+h(x)^2)$$
$$= \alpha(x^2 + (\beta-2\alpha)x^3 + (9\alpha^2 - 7\alpha\beta+\beta^2)x^4) + O(x^5)$$
$$= \alpha x^2 + \alpha(\beta-2\alpha)x^3 + \alpha(9\alpha^2-7\alpha\beta+\beta^2)x^4 + O(x^5)$$

If $\alpha\neq 0$, then $\dot x = \alpha x^2 + \alpha(\beta-2\alpha)x^3 + O(x^4)$

#2.2.5.

The attempt using Taylor series to compute h may fail, if W^c is not analytic at origin.

Example 2.2.6 Consider $\begin{cases}\dot x = -x^3 \\ \dot y = -y+x^2\end{cases}$ around $(0,0)$.

$J = Df(0,0) = \begin{pmatrix}0&0\\0&-1\end{pmatrix}$ already block form, so $f(x,y) = -x^3$, $g(x,y)=x^2$.

Let $h(x) = ax^2+bx^3+cx^4+\dots$, then (N) becomes: $B=0$, $C=-1$

$$\begin{cases} h'(x)(-x^3) + h(x) - x^2 = 0 \\ h(0)=0,\ h'(0)=0\end{cases} \Rightarrow -x^3(2ax+3bx^2\overset{+4cx^3}{+}\dots) + ax^2 + bx^3 \overset{+cx^4}{+}\dots - x^2 = 0$$

x^2-coeff: $a-1=0\Rightarrow a=1$

x^3-coeff: $b=0$

x^4-coeff: $-2a+c=0 \Rightarrow c=2a=2$

$\Rightarrow h(x) = x^2+2x^4+\dots$

but $W^c = \{(x,y): y=0\}$

So $h(x)$ must be 0!

$\Rightarrow \dot x = -x^3$ always!

#2.2.6