Remark 1.9.5 (i) Morse-Smale $\Longrightarrow$ s. stable, and $\not\Longleftarrow$

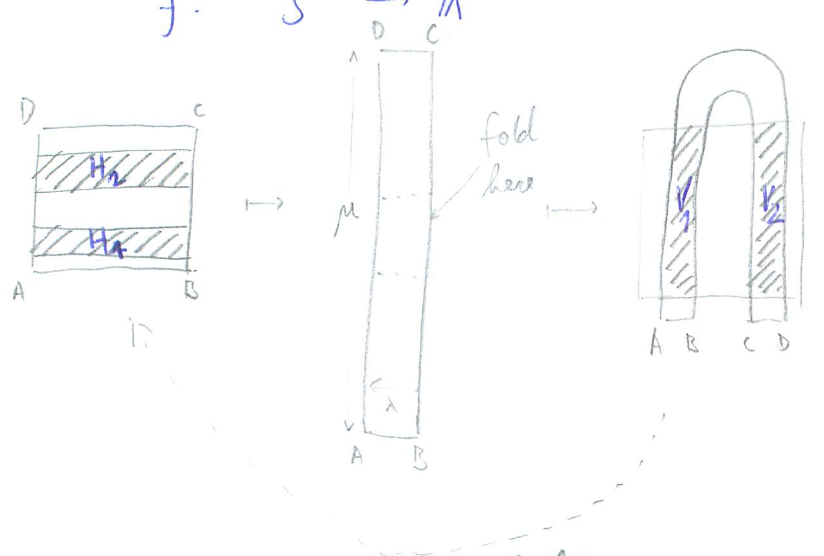
(ii) Morse-Smale systems are not dense in $\mathcal{F}^r(M^m)$
S. stable " are not dense in $\mathcal{F}^r(M^m)$ in general, if $m>2$.

Example 1.9.6. (The Smale Horseshoe)

Let $S=[0,1]\times[0,1]$ be the unit square in $\mathbb{R}^2$. Define

$f: S \to \mathbb{R}^2$

$0<\lambda<\frac{1}{2}$
$\mu>2$



$f^{-1}(S\cap f(S))$

$f: f^{-1}(S\cap f(S)) \to S\cap f(S)$

Jacobian of $f|_{H_1}: H_1 \to V_1$ is $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$

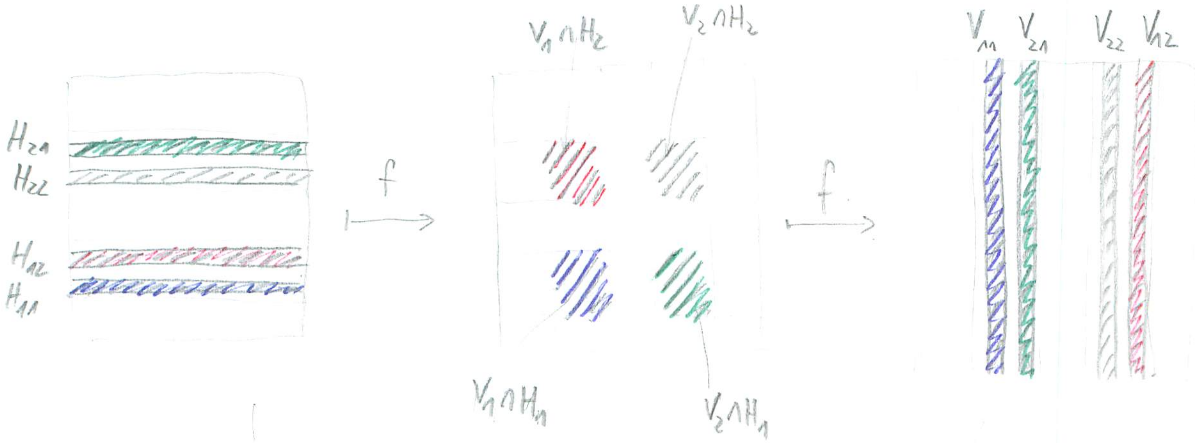
" " $f|_{H_2}: H_2 \to V_2$ is $\begin{pmatrix} -\lambda & 0 \\ 0 & -\mu \end{pmatrix}$

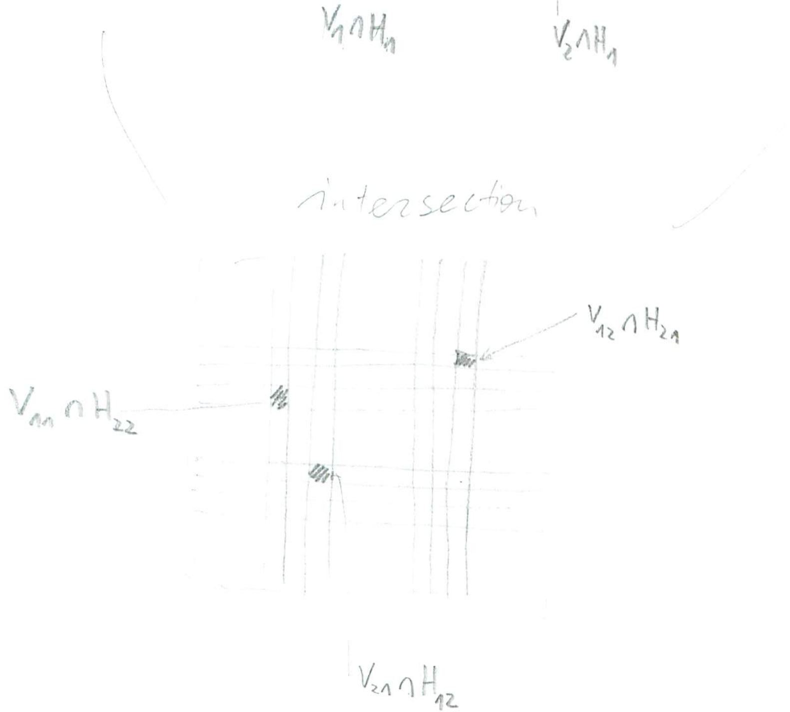
Consider the iteration of f: $\{f^i : -\infty<i<\infty\}$ and those points that do not leave S for all time:

$\Lambda = \{x : f^i(x) \in S \ \forall_{-\infty<i<\infty}\}$

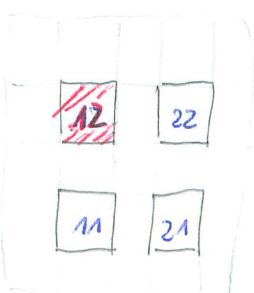
e.g.

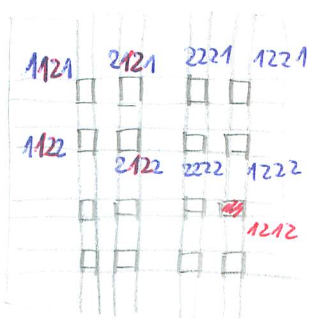
$\Lambda^1 = \{x : f^i(x) \in S : i=\pm 1, 0\}$





Using indices of H_i's and V_j's : we have





$\bigcap_{n=-1}^{1} f^n(S)$ $\qquad \bigcap_{n=-2}^{2} f^n(S)$ $\qquad \ldots \ldots \longrightarrow \Lambda$.

<u>Theorem 1.9.6(a)</u> There is a 1-1 correspondence ϕ between Λ and the set Σ of bi-infinite sequences of two symbols such that the sequence $b = \phi(f(x))$ is obtained from the sequence $a = \phi(x)$ by left shifting. $b_i = a_{i+1}$. The set Σ has a metric defined by

$$d(a,b) = \sum_{i=-\infty}^{\infty} \delta_i 2^{-|i|}, \quad \delta_i = \begin{cases} 0 & \text{if } a_i = b_i \\ 1 & \text{if } a_i \neq b_i \end{cases}$$

The map ϕ is a homeomorphism from Λ to Σ endowed with this metric.

Pf: W.L.O.G. $\Sigma = \left\{ a = \{a_i\}_{i=-\infty}^{\infty} : a_i \in \{1,2\} \right\}$ two symbols 1 and 2.

Define $\phi : \Lambda \longrightarrow \Sigma$ by

$$x \longmapsto \phi(x)$$

where $\phi(x) = \{a_i\}_{i=-\infty}^{\infty}$, with $f^i(x) \in H_{a_i}$,

In words, $x \in \Lambda \Leftrightarrow f^i(x)$ is in $H_1 \cup H_2$ for all i,

we associate to x the sequence to tell us which of H_1 and H_2 contains $f^i(x)$.

x :	...	$f^{-2}(x)$	$f^{-1}(x)$	$f^0(x)$	$f(x)$	$f^2(x)$	...	
				✓		✓		H_1
								or
		✓	✓		✓			H_2

$\rightsquigarrow$ $\phi(x) = \ldots 2\,2\,1\,2\,1 \ldots$

Since $f^{i+1}(x) = f^i(f(x))$, it follows that $\phi(f(x))$ is the left shift of $\phi(x)$.

One verifies that ϕ is 1-1, conts, and onto.

#.

With Theorem 1.9.6.(a), the analysis of Horseshoe's dynamics is reduced to that of "shifting operator" on Σ:

$$\sigma: \Sigma \to \Sigma$$
$$a \mapsto \sigma(a) =: b, \quad \text{with} \quad b_i = a_{i+1}.$$

By Thm 1.9.6.(a), we have

$$\phi \circ \left(f|_\Lambda\right) = \sigma \circ \phi, \quad \text{i.e.} \quad f|_\Lambda = \phi^{-1} \circ \sigma \circ \phi$$

thus $f^n|_\Lambda = \phi^{-1} \circ \sigma^n \circ \phi$, i.e. ϕ maps orbits of f in Λ to orbits of σ in Σ. As example,

$$\#\text{ of per. Orbits of period } n \text{ for } f = \#\text{ of per. Orbits of period } n \text{ for } \sigma$$
$$= 2^n$$

Theorem 1.9.6.(b) The horseshoe map f has an invariant Cantor set Λ such that

(a) Λ contains a countable set of periodic orbits of arbitrary long periods.

(b) Λ contains an uncountable set of bounded nonper. ~~points~~ motions.

(c). Λ contains a dense orbit.

Moreover, any sufficiently C^1-close map $\tilde{f}$ has an inv. Cantor Set $\tilde{\Lambda}$ with $\tilde{f}|_{\tilde{\Lambda}}$ topo. equiv. to $f|_\Lambda$. #.

2. Bifurcation (local theory)

2.1. Bifurcation Problems

Consider

$$(*) \qquad \dot{x} = f(\mu, x) \quad , \quad x \in \mathbb{R}^n, \ \mu \in \mathbb{R}^k, \ f \text{ sufficiently smooth.}$$

φ_μ associated flow, for $\mu \in \mathbb{R}^k$

Several possible definition of bifurcation value of μ:

Def 2.1.a A value μ_0 is a bifurcation value for $(*)$, if the flow of $(*)$ at $\mu = \mu_0$ is not structurally stable.

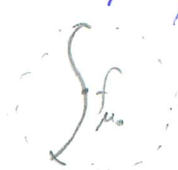
Def 2.1.a' Given an equivalence relation "$\sim$" on all flows, a value μ_0 is a bifurcation value for $(*)$, if $\forall_{\varepsilon > 0} \ \exists_\mu$ with $|\mu - \mu_0| < \varepsilon$ s.t. $\varphi_\mu \nsim \varphi_{\mu_0}$

Rmk:

(i) If we take the equivalence relation $\sim$ to be the topological equivalence, then Def 2.1.a' gives Def. 2.1.a.

(ii) Def. 2.1.a is too general, it means one needs to study all flows in arbitrary nbhd to decide whether μ_0 is a bf. value.

Def. 2.1.b A value μ_0 is a bifurcation value for $(*)$, if in every nbhd of μ_0, there exists μ such that φ_μ is not topo. equiv. to φ_{μ_0}.



Beispiel 2.1.2 $\quad \dot{x} = -(\mu^2 x + x^3)$,

$\mu_0 = 0$ is a BF value in the sense of Def. 2.1.a,
but NOT a BF value in the sense of Def. 2.1.b. (HW)